We write the number three thousand six hundred fifty seven as
\[ 3657 = 3 \cdot 1000 + 6 \cdot 100 + 5 \cdot 10 + 7 = 3 \cdot 10^3 + 6 \cdot 10^2 + 5 \cdot 10^1 + 7 \cdot 10^0. \]
Is there anything special about using 10? From a mathematical point of view, the answer is no.

Proposition: Fix some \( n \) (called the base). We can write any number uniquely in the form
\[ a_k n^k + a_{k-1} n^{k-1} + \cdots + a_2 n^2 + a_1 n + a_0 n^0 \]
where each \( a_i \) takes values from 0 to \( n - 1 \).
To avoid ambiguity, we will write the base as a subscript. For example, in base 6, the number fifty \((50_{10})\) is
\[ 1 \cdot 6^2 + 0 \cdot 6 + 2 \cdot 6^0 = 122_6. \]

1) Convert the following numbers to base 10.
   a) \( 1232_4 = 110_{10} \)
   b) \( 10120_3 = 96_3 \)
   c) \( 723_9 = 588_{10} \)
   d) \( 100111_2 = 39_{10} \)
   e) \( 1232_5 = 192_{10} \)
2) Convert the number \( 98_{10} \) into: (can you come up with a fast way to do this?)
   a) base 7: \( 200_7 \)
   b) base 4: \( 1202_4 \)
   c) base 3: \( 10122_3 \)
   d) base 2: \( 1100010_2 \)
3) Write out multiplication tables for base 2 and base 3.

<table>
<thead>
<tr>
<th>Base 2</th>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Base 3</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>11</td>
</tr>
</tbody>
</table>

We can add/multiply numbers using the same methods we know for base 10.

4) Calculate (in the same base as given):
   a) \( 11010_2 + 10011_2 = 101101_2 \)
   b) \( 21012_3 + 121201_3 = 212220_3 \)
c) \(1001_2 \cdot 101_2 = 101101_2\)

d) \(2111_3 \cdot 122_3 = 1120012_3\)

5) Is it possible that the following statements are true in some number base system? Is so, what base?

a) \(3 \cdot 4 = 10\): base 12

b) both \(3 + 4 = 10\) and \(3 \cdot 4 = 15\): base 7.

c) both \(2 + 3 = 5\) and \(2 \cdot 3 = 11\):

Not possible. First equation needs base \(\geq 6\), while second needs base 5.

6) State and prove a condition (involving the representation of a number) which allows us to determine whether the number is even or odd:

a) in the base 3 system.

b) in the base \(n\) system.

If \(n\) is even: a number is odd if and only if the last digit is odd in the representation in base \(n\).

If \(n\) is odd: a number is odd if and only if there are an odd number of odd digits in the representation in base \(n\).

7) An evil king wrote three secret two-digit numbers \(a, b, c\). A handsome prince must name three numbers \(X, Y, Z\), after which the king will tell him the sum \(aX + bY + cZ\). The prince must then name all three of the King’s numbers, or he will be executed. Help out the prince!

Check that \(X = 100^2, Y = 100, Z = 1\) works.

8)* Prove the proposition stated at the beginning of the handout.

9)* Prove that from the set \(0, 1, 2, \ldots, 3^k - 1\) one can choose \(2^k\) numbers so that none of them can be represented as the arithmetic mean of some pair of the chosen numbers.

Hint: Work in base 3. The \(2^k\) numbers that we want are those that only have 0 and 1 in their base 3 representation.

Some problems are taken from:

- D. Fomin, S. Genkin, I. Itenberg “Mathematical Circles (Russian Experience)”