Enumerative combinatorics is a field of mathematics that deals with counting the number of objects satisfying some combinatorial description.

1. Catalan Numbers

1.1. Exercise. A monotonic path is a finite sequence of ‘up’ and ‘right’ steps of unit length. Count the number of monotonic paths from \((0,0)\) to \((n,m)\).

1.2. Exercise. Consider monotonic paths from \((0,0)\) to \((n,n)\) which cross above the diagonal, i.e., those that enter the region \(y > x\). Count the number of such paths by establishing a bijection with monotonic paths from \((0,0)\) to \((n-1,n+1)\).

1.3. Exercise. Count the number of monotonic paths from \((0,0)\) to \((n,n)\) which do not cross above the diagonal, i.e., those that stay in the region \(y \leq x\).

1.4. Exercise. Write the answer from the previous exercise as a fraction of a binomial coefficient. This is known as the Catalan number \(C_n\).

1.5. Exercise. Let \(\nearrow = (1,1)\) and \(\searrow = (1,-1)\) be vectors in \(\mathbb{Z}^2\). Let

\[
S = \left\{ (v_1, v_2, \ldots, v_{2n}) \in \{\nearrow, \searrow\}^{2n} : \sum_{i=1}^{2n} v_i = (2n,0) \right\}
\]

be the collection of all sequences of length \(2n\) with the same number of \(\nearrow\) as \(\searrow\). Count the number \(|S|\) of such sequences.

1.6. Exercise. Let \(B\) be the sequences \((v_1, \ldots, v_{2n}) \in S\) such that each partial sum \(s_t = \sum_{i=1}^{t} v_i\) has non-negative \(y\)-coordinate for \(t \in [2n] = \{1, 2, \ldots, 2n\}\). These are known as ballot sequences—a way for \(2n\) people to cast yes/no votes one at a time so that there are always at least as many “yes” as “no” votes, with a tie as the end result. Count the number \(|B|\) of ballot sequences.

1.7. Exercise. Prepend \(v_0 = \nearrow\) in front of every sequence in both \(S\) and \(B\), and call the resulting sequences augmented and denote the sets \(\overline{S}\) and \(\overline{B}\), respectively. For each augmented sequence \((v_0, v_1, \ldots, v_{2n}) \in \overline{S}\), show that there is a unique \(r \in [0,2n] = \{0, 1, \ldots, 2n\}\) such that \((v_0+r, v_1+r, \ldots, v_{2n+r})\) is an augmented ballot sequence in \(\overline{B}\), where the indices are read modulo \(2n+1\). Call the map \(f : \overline{S} \rightarrow \overline{B}\) that sends the augmented sequence to its corresponding augmented ballot sequence.

1.8. Exercise. Show that \(f\) is surjective and that the fibers under \(f\) are all of the same size, and calculate this size. That is, calculate \(|f^{-1}(b)|\) for \(b \in B\).

1.9. Exercise. Establish a linear relationship between \(|B|\) and \(|S|\).

1.10. Exercise. Conclude a second proof of the expression of the Catalan number in \([1.4]\) without using subtraction, but using division instead.