Homogeneous Coordinates

The homogeneous coordinates of a point \((X, Y) \in \mathbb{R}^2\) are all the real ordered triples \((Xz, Yz, z)\) with \(z \neq 0\), i.e. all triples \((x, y, z)\) such that \(x/z = X\) and \(y/z = Y\). If we take \((X, Y)\) to be coordinates in the plane \(z = 1\), then these are the coordinates of points on the line from the origin to \((X, Y)\). We denote homogeneous coordinates by \((x : y : z)\).

Thus homogeneous coordinates give a one-to-one correspondence between points \((X, Y) \in \mathbb{R}^2\) and nonhorizontal lines through the origin in \(\mathbb{R}^3\). The horizontal lines, which have homogeneous coordinates of the form \((x, y, 0)\), naturally correspond to points at infinity.

1. Which of the following homogeneous coordinates represent the same points? Which represent points at infinity?

\((-1 : -1 : -1)\) \((2 : 3 : 5)\) \((0 : 0 : 1)\) \((-1 : 0 : 0)\) \((2 : -3 : 0)\) \((2 : 2 : 2)\)
\((4 : -6 : 0)\) \((1 : \frac{3}{2} : \frac{5}{2})\) \((6 : 6 : 6)\) \((0 : 1 : 3)\) \((0 : 0 : 3)\) \((10000 : 10000 : 10000)\)

2. We say that \(p(x, y, z)\) is a homogeneous polynomial of degree \(m\) if

\(\bullet\) \(p\) is a sum of terms of the form \(a_{jkl}x^jy^kz^\ell\), where \(j + k + \ell = m\)

\(\text{OR}\)

\(\bullet\) For any \(t\), \(p(tx, ty, tz) = t^mp(x, y, z)\).

(See problem (2b) below.)

We make the analogous definition in one, two, or any other number of variables.

(a) Identify the homogeneous polynomials and give their degree:
\[xyz, x^2 - y^2, 2x + 2y - z, x^2 + 3y^2 + 5z^2, xy - yz + 3xz, xy^3 - x^4 + 3xy^2z - 6xz^2, y - x^2, 3xy + 50y^2 - z, x^{10} + y^5z^5 - 2x^3z^7\]

(b) Prove that the two definitions above are equivalent.
3. An *algebraic curve* is a curve defined by an equation \( p(X, Y) = 0 \), where \( p \) is a polynomial in \( X, Y \). Examples:

If \( p \) is a polynomial of degree \( n \), we can write the equation \( p(X, Y) = 0 \) as \( p\left(\frac{x}{z}, \frac{y}{z}\right) = 0 \) when \( z \neq 0 \), and then extend to all values of \( z \) by multiplying through by \( z^n \):

\[
\bar{p}(x, y, z) = 0, \quad \text{where} \quad \bar{p}(x, y, z) = z^n p\left(\frac{x}{z}, \frac{y}{z}\right).
\]

**Example:** The equation \( X^3 + XY - Y = 0 \) becomes \( \left(\frac{x}{z}\right)^3 + \left(\frac{x}{z}\right) \left(\frac{y}{z}\right) - \frac{y}{z} = 0 \) becomes \( x^3 + xyz - yz^2 = 0 \).

Having done this, we can include *points at infinity* which are solutions of the extended polynomial equation satisfying \( z = 0 \).

(a) For each curve, find how many points at infinity there are for \( p(X, Y) = 0 \)
\[
X + 2Y \quad X^2 - Y^2 \quad X^2 + Y^2 \quad Y - X^2
\]
(b) How many points of intersection (including points at infinity) are there between:
   i. \( y = 0 \) and \( y = x^2 + 1 \) ii. \( x + y = 0 \) and \( x - y = 0 \) iii. \( xy = 1 \) and \( y = -x \)
   iv. \( 2x + y = 0 \) and \( 2x + y = 1 \)