Continued Fractions

A finite continued fraction is an expression of the form

\[ a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}} \]

where we will assume always that \( a_1, \ldots, a_n \) are positive integers. We often use the more compact notation \([a_0, a_1, \ldots, a_n]\).

To compute the continued fraction expansion of a number, for example \(3 \frac{2}{3}\), we use the following recursive procedure:

- Write \(3 \frac{2}{3} = 3 + \frac{2}{3}\), separating out the largest whole number possible.
- Rewrite this as \(3 + \frac{1}{\frac{3}{2}}\), expressing the non-whole number part \(\frac{2}{3}\) as 1 over its reciprocal.
- Repeat the procedure on the denominator \(\frac{3}{2}\).

If there is nothing left over after we take away the largest whole number, the procedure terminates. Here is the procedure carried out on \(3 \frac{2}{3}\):

\[
3 \frac{2}{3} = 3 + \frac{2}{3} = 3 + \frac{1}{\frac{3}{2}} = 3 + \frac{1}{1 + \frac{1}{2}}.
\]
Exercises

1. Compute a continued fraction expansion for each of the following numbers:
   (a) $\frac{5}{12}$    (b) $\frac{5}{3}$    (c) $\frac{33}{23}$    (d) $\frac{37}{31}$

2. For each continued fraction, write the corresponding number as a reduced fraction:
   (a) $[2, 3, 2]$    (b) $[1, 4, 6, 4]$    (c) $[2, 3, 2, 3]$    (d) $[9, 12, 21, 2]$
Infinite Continued Fractions

The procedure outlined in the previous section may be carried out for any number $x$: At each step we subtract the largest whole number that we can, keeping the result nonnegative. Then we invert the remainder and repeat.

Example

If $x = \sqrt{2} \approx 1.414\ldots$, we have

\[
\sqrt{2} = 1 + \frac{1}{1 + \sqrt{2}}
\]

\[
= 1 + \frac{1}{1 + \left(1 + \frac{1}{1 + \sqrt{2}}\right)}
\]

\[
= 1 + \frac{1}{2 + \frac{1}{1 + \sqrt{2}}}
\]

\[
= \ldots
\]

Exercise

1. Find continued fraction expansions for each irrational number.
   
   (a) $\sqrt{3}$   (b) $\sqrt{4}$ :)   (c) $\sqrt{5}$   (d) $\sqrt{6}$   \ldots (as many as you can!)