Probability via problem solving

Intuitive understanding

Topics to discuss: Likelihood vs probability,"random" experiments (coin toss, card deck, etc), (counter)intuitive facts, simple calculations.

Problems:

(1) After tossing 6 heads in a row, which is more likely to get next: heads or tails?
(2) How likely is to turn over half of a deck of 52 cards without seeing a king?
(3) Pick a random number from $\{1,\ldots,20\}$. How likely is it that 3 divides this number?
(4) (Hard) Alice and Bob alternate in tossing a coin. Alice is betting \$1 on heads, Bob is betting \$1 on tails. They initially start with the same amount of money. How likely is it that Alice will stay ahead of Bob for 10 rounds of the game? (This is the basis of a *reflection principle*.)
(5) (Hard) For a (random) group of 22 people, how likely is it that (some) two of them will have birthday on the same day? (This is the *Birthday problem*.)

Formal approach to probability

Topics to discuss: Disjoint (mutually exclusive) events, complementarity, monotonicity, inclusion-exclusion, independent events, dependence, etc.

Problems:

(6) What is the probability that a randomly chosen card from a deck is a queen or a heart.
(7) A wife and a husband are seated randomly at a round-table dinner of 12 friends. How likely is is that they sit next to each other?
(8) How likely is it to toss a coin 10 times and see no more than 2 heads?
(9) For a random number from $\{1,\ldots,100\}$, how likely is it divisible by 4 or 5?
(10) You are sequentially dealt 5 cards from the deck. How likely is it that they come in increasing value (e.g., $2\spadesuit$ followed by $5\clubsuit$ followed by $5\heartsuit$ followed by $7\diamondsuit$, etc.).
(11) How likely is it that you need to toss a coin 20 times to see the 3rd heads come up? (This is an example for the *Hypergeometric distribution*.)
(12) 50 red and 50 blue marbles are to be put into two urns whichever way you choose. Then a random urn is picked and a random ball is drawn. What is the division that maximizes the probability of drawing a red marble?

Geometric aspects

Topics to discuss: Continuous distributions, computing probability by geometric means, applications.

Problems:

(13) A random point X is chosen between 0 and 1. Write X in decimal expansion as $0.x_1x_2x_3\ldots$. What is the probability that there is no 5's among the first 10 digits? And what is the probability that there is no 5 at all?

(14) Show that a random point X takes value $\frac{1}{2}$ (or any other given value) with probability zero.

(15) (Hard) Show that the point takes any rational value with probability zero.

(16) Draw a curve above the x axis which lies entirely in the square with corners $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$. The curve separates the "bottom" of the square from the "top." What is the probability that a random point in the square lies "below" this curve? Experiment with simple examples and then draw a general conclusion. (This is the basis of a *Monte Carlo method*.)

Further challenging problems

(17) A beauty queen is being proposed to by 40 men. She arranges an interview with each of them to find out his ranking — which is a random number between 0 and 100. (No ties allowed.) The problem is that once she rejects a suitor, he won't come back again. Here is an algorithm she chooses: She sees a group of k men first (with $k < 40$), rejects them all but records the best score she has seen. Call this score s. Then she calls the remaining ones, one after another, and simply picks the first one who has score better than s — or the last one, if none else is left. How likely does she pick the best man of all 40 suitors? (This is the *Secretary problem*.)

(18) Ten men are leaving a bar at night and the bartender returns them their hats. They are very happy to pick any hat he gives them. How likely is it that no man has his own hat the next morning?

(19) In the country of Shyland—where people are too shy to talk—a group of 145 men are boarding a plane. The first one to enter the cabin finds out that he has misplaced his boarding pass and so he just picks a random seat. The next one goes to his assigned seat, but if that is taken (by the first man) he rather picks one at random. And this goes on and on until the last person is boarding the plane. The flight is full so there is only one seat left; how likely is it the last person's assigned seat?

(20) A needle of length 1in is dropped on the floor that has parallel lines 1in apart. What is the probability that the needle will intersect one of the lines. (This is the *Buffon needle problem*.)