PERMUTATIONS - II

JUNIOR CIRCLE 05/01/2011

(1) Play the following game with your partner several times:
- Take 5 cards with numbers $1, 2, 3, 4, 5$ written on them;
- Mix the order of the cards and put them on the table in the new order;
- Ask your partner to return the cards to the original order $(1, 2, 3, 4, 5)$ by repeating the following operation:
  - *switch card in position* 1 *with a card in another position;*

  Do you think you can always return to the original order this way?

(2) Play the following game with your partner several times:
- Take 5 cards with numbers $1, 2, 3, 4, 5$ written on them;
- Mix the order of the cards and put them on the table in the new order;
- Ask your partner to return the cards to the original order $(1, 2, 3, 4, 5)$ by repeating the following operation:
  - *switch card with number* 1 *written on it with any other card*;

  Do you think you can always return to the original order this way?

  Compare the games in this and previous problems.

(3) Let's look at how the order of the cards on the table corresponds to a permutation.

(a) Suppose that we have the following permutation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 3 & 1 & 2 \end{pmatrix}.$$

This means that

- card $\boxed{1} \to \boxed{}$;
- card $\boxed{2} \to \boxed{}$;
- card $\boxed{3} \to \boxed{}$;
- card $\boxed{4} \to \boxed{}$;

Write down the order of the cards on the table that corresponds to this permutation:

$\boxed{}$ $\boxed{}$ $\boxed{}$ $\boxed{}$

(b) Suppose the cards are put on the table in the following order:

$\boxed{2}$ $\boxed{3}$ $\boxed{1}$ $\boxed{4}$

This means that

- card $\boxed{1} \to \boxed{}$;
- card $\boxed{2} \to \boxed{}$;
- card $\boxed{3} \to \boxed{}$;
- card $\boxed{4} \to \boxed{}$;

Write down the permutation that corresponds to this order of the cards:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ & & & \end{pmatrix}.$$

(4) Given cards in the order $(3, 2, 4, 1)$ How can we get back to the original position $(1, 2, 3, 4)$ by performing several switches with position 1 (i.e., with the card that is on the left)?

Model this with the following:

- First, switch cards in positions $\boxed{1}$ and $\boxed{}$;
- Second, switch cards in positions $\boxed{1}$ and $\boxed{}$;
- Third, switch cards in positions $\boxed{1}$ and $\boxed{}$;
- Forth, switch cards in positions $\boxed{1}$ and $\boxed{}$;
- Fifth, switch cards in positions $\boxed{1}$ and $\boxed{}$;

  Note: You may not need to use all 5 of these steps, you can use less and still put the numbers into the original order.

(5) Given cards in the order $(3, 2, 1, 5, 4)$, how can we get back to the original position $(1, 2, 3, 4, 5)$ by performing several switches with position 1?

- Model this with the following:
  - First, switch cards in positions $\boxed{1}$ and $\boxed{}$;
  - Second, switch cards in positions $\boxed{1}$ and $\boxed{}$;
  - Third, switch cards in positions $\boxed{1}$ and $\boxed{}$;
  - Forth, switch cards in positions $\boxed{1}$ and $\boxed{}$;
  - Fifth, switch cards in positions $\boxed{1}$ and $\boxed{}$;
  - Sixth, switch cards in positions $\boxed{1}$ and $\boxed{}$;

  Note: You may not need to use all 6 of these steps, you can you less and still put it into the original position.

(6) Find the result of performing two permutations in a row:

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix}$$

- Model this with cards to figure out where the positions move after the two switches:

$$1 \to \boxed{} \to \boxed{}$$

$$2 \to \boxed{} \to \boxed{}$$

$$3 \to \boxed{} \to \boxed{}$$

(7) Find the result of performing two permutations in a row:

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix}$$

- Compare the results from 6 and 7. Are they the same? Why or why not? Does the order matter in which you perform these two permutations matter?

(8) Find the result of permorming two permutations in a row:

(a)

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 & 4 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ & & & \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 4 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ & & & \end{pmatrix}$$

(c) Compare the results. Are they the same? Why or why not?

(d) Does the order in which you perform these two permutations matter?

(e) What is the difference between this problem and the previous one?

(9) Give a permutation, find another one so that when the first is followed by the second you return to the original order:

(a)
$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 2 & 1 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 \end{pmatrix}$$

(b)
$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 3 & 1 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 \end{pmatrix}$$

(c)
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ & & & \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

(d)
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 1 & 5 & 3 & 4 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & & & & \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

(e) Do you think you can always "undo" a permutation in this way?

(10) Write down all the possible permutations of 2 objects:

$$\begin{pmatrix} 1 & 2 \\ \downarrow & \downarrow \\ & \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ \downarrow & \downarrow \\ & \end{pmatrix}$$

(11) Write down all the possible permutations for 3 objects:

$$\begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ \downarrow & \downarrow & \downarrow \\ & & \end{pmatrix}.$$

(12) In a permutation of 3 objects,

(a) How many choices do we have for the new position of object 1?

(b) After this choice is made, how many choices do we have fore the new position of object 2?

(c) Finally, after both of these choices have been made, how many choices do we have for the new position of object 3?

(d) Can you figure out the total number of permutations of 3 objects? Does your answer agree with what you have done in the previous problem?

(13) *(Challenge!)* Can you figure out (without writing them down!) the number of permutations of 4 objects? Explain how you got your answer.

(14) How many different ways can 3 people (Alice, Bob and Cy) stand in a row?

(15) How many different ways can 3 people sit around the table? (*Note:* two positions where the three people are just shifted by one place are considered to be the same)

(16) Are you answers in #14 and #15 the same? Why or why not?

(17) Three people (Andy, Brian and Cole) are competing in a race. How many different ways can the prizes for 1st, 2nd and 3rd place be distributed? (There are *no ties*)