15 Binary magic

Materials for the lesson and homework: a couple of regular pencils, an eraser, a pencil sharpener; three standard size flash cards, magic cards from the end of the book.

There are seven sheets with magic cards at the end of the book. The first sheet has six smaller size cards. These are for a student. If you are a homeschooling family, this is all you need. Please cut the sheet out of the book and further cut it into six cards. If you teach in a classroom setting, you will also need six larger size cards printed on the subsequent sheets. Please cut them out as well in preparation for the lesson.

Warm-up

Problem 15.1 *In a family, a boy has as many sisters as brothers while a girl has twice as many brothers as sisters. How many boys and girls are there in the family?*

There are ____ boys and ____ girls in the family.

15.1 Lesson

We have solved a problem very similar to the problem 15.2 below before. We refresh the material here because we put it to good use in what follows.

Problem 15.2 *Fill out the following table to find the binary decomposition of the numbers 1 through 7.*

studied number	*basic binary numbers*						
	64	**32**	**16**	**8**	**4**	**2**	**1**
1							
2							
3							
4							
5							
6							
7							

Take three standard size flash cards. Write the first basic binary number, 1, in the upper-left corner of one of the cards. We will call the card the *first*. Write the second basic binary number, 2, in the upper-left corner of the next card. We will call this card the *second*.

Write the third basic binary number, 4, in the upper-left corner of the remaining card. We will call this card the *third*.

Problem 15.3

- *On the first flash card, write down all the integers 1 through 7 such that the number* 1 *is a part of their decomposition into a sum of basic binary numbers. Use the table from problem 15.2 if needed.*

- *Are the numbers on the first flash card odd, even, or both? Circle the correct answer.*

Odd *Even* *Both*

- *Does every odd number have 1 as a part of its binary decomposition? Circle the correct answer.*

Yes *No*

Explain your choice.

Problem 15.4 *On the second flash card, write down all the integers 1 through 7 such that the number* 2 *is a part of their decomposition into a sum of basic binary numbers. Use the table from problem 15.2 if needed.*

Problem 15.5 *On the third flash card, write down all the integers 1 through 7 such that the number* 4 *is a part of their decomposition into a sum of basic binary numbers. Use the table from problem 15.2 if needed.*

Problem 15.6

- *A number appears on the first flash card only. What is the number?*

  The number is _____ .

- *A number appears on the third flash card only. What is the number?*

  The number is _____ .

- *A number appears on the first and third flash cards only. What is the number?*

  The number is _____ .

- *A number appears on all the three cards. What is the number?*

  The number is _____ .

Solve the following problem without looking at the cards.

Problem 15.7

- *A number appears on the first and second flash cards only. What is the number?*

  The number is _____ .

- *Explain your answer.*

Problem 15.8 *Use the three flash cards to design a trick similar to the one you have seen in class. Practice the trick with your neighbor.*

Now you are ready to understand the original magic trick!

Similar to the above, we call the magic card from the end of the book that has the first basic binary number, 1, in the upper-left corner the *first*. We call the magic card with the second basic binary number, 2, in the upper-left corner the *second*. We call the magic card with the third basic binary number, 4, in the upper-left corner the *third*. The *fourth*, *fifth*, and *sixth* magic cards have the basic binary numbers 8, 16, and 32 in the the upper-left corners respectively.

Problem 15.9

• *Are the numbers on the first magic card odd, even, or both? Circle the correct answer.*

Odd *Even* *Both*

• *Does every odd number have 1 as a part of its binary decomposition? Circle the correct answer.*

Yes *No*

Explain your choice.

• *Do all the numbers on the first magic card have 1 as a part of their binary decomposition? Circle the correct answer.*

Yes *No*

• *Do all the odd numbers in between 1 and 63 appear on the first magic card? Circle the correct answer.*

Yes *No*

- *Is it correct to say that the first magic card shows all the integers from 1 to 63 that have the basic binary number 1 as a part of their decomposition into a sum of basic binaries? Circle the correct answer.*

Yes *No*

It's not hard to see a pattern on the second magic card: 2 and 3 are there, 4 and 5 are missing, 6 and 7 are there, 8 and 9 are missing, and so forth. To figure out what the pattern tells us, let us break down the first fourteen numbers from the card into a sum of basic binaries. On the left-hand side of the table below, we use boldface font for the numbers that are on the second magic card and regular font for the numbers that are not. We also use the boldface font to highlight the basic binary 2 in the decompositions on the right-hand side of the table.

$$
\begin{array}{rcl}
\mathbf{2} & = & \mathbf{2} \\
\mathbf{3} & = & \mathbf{2} + 1 \\
4 & = & 4 \\
5 & = & 4 + 1 \\
\mathbf{6} & = & 4 + \mathbf{2} \\
\mathbf{7} & = & 4 + \mathbf{2} + 1 \\
8 & = & 8 \\
9 & = & 8 + 1 \\
\mathbf{10} & = & 8 + \mathbf{2} \\
\mathbf{11} & = & 8 + \mathbf{2} + 1 \\
12 & = & 8 + 4 \\
13 & = & 8 + 4 + 1 \\
\mathbf{14} & = & 8 + 4 + \mathbf{2} \\
\mathbf{15} & = & 8 + 4 + \mathbf{2} + 1
\end{array}
$$

Problem 15.10

- *Explain the pattern.*

- *Do all the numbers on the second magic card have the basic binary number 2 as a part of their decomposition into a sum of basic binaries? Circle the correct answer.*

Yes *No*

- *Is it correct to say that the second magic card shows all the integers from 1 to 63 that have the basic binary number 2 as a part of their decomposition into a sum of basic binaries? Circle the correct answer.*

Yes *No*

Explain your choice.

Problem 15.11 *Find the pattern on the third magic card. Hint: pay attention to the basic binary number 4.*

Problem 15.12

- *Explain the pattern on the third magic card.*

- *Do all the numbers on the third magic card have the basic binary number 4 as a part of their decomposition into a sum of basic binaries? Circle the correct answer.*

Yes *No*

- *Is it correct to say that the third magic card shows all the integers from 1 to 63 that have the basic binary number 4 as a part of their decomposition into a sum of basic binaries? Circle the correct answer.*

Yes *No*

Explain your choice.

Let us now look at the sixth magic card.

Problem 15.13

- *Is it correct to say that all the integers from 32 to 63 appear on the sixth magic card? Circle the correct answer.*

Yes *No*

- *Is it correct to say that the basic binary number 32 is a part of the decomposition of every integer from 32 to 63 into a sum of basic binaries? Circle the correct answer.*

Yes *No*

Explain your choice.

- *Is it correct to say that the sixth magic card shows all the integers from 1 to 63 that have the basic binary number 32 as a part of their decomposition into a sum of basic binaries? Circle the correct answer.*

Yes *No*

Make a guess to answer the following.

Question 15.1 *What kind of numbers are there on the fourth magic card? On the fifth?*

Discuss question 15.1 with students. Make sure they understand that the fourth magic card lists all the integers from 1 to 63 that have 8 as a part of their decomposition into the sum of basic binaries. Similarly, the fifth magic card lists all the integers from 1 to 63 that have 16 as a part of their binary decomposition.

Problem 15.14

- *A number appears on the first magic card only. What is the number?*

The number is _____ .

- *A number appears on the second magic card only. What is the number?*

The number is _____ .

- *A number appears on the fourth magic card only. What is the number?*

  The number is _____ .

- *A number appears on the first and second magic cards only. What is the number?*

  The number is _____ .

- *A number appears on the third and fifth magic cards only. What is the number?*

  The number is _____ .

- *A number appears on the first, second, and sixth magic cards only. What is the number?*

  The number is _____ .

- *A number appears on the third, fourth, and fifth magic cards only. What is the number?*

  The number is _____ .

- *A number appears on all the magic cards. What is the number?*

  The number is _____ .

- *A number appears on all the magic cards except for the second. What is the number?*

  The number is _____ .

Question 15.2 *How does the trick work?*

Let us call the flash cards with integers 1 through 7 we used at the beginning of the lesson *small*. Let us call the magic cards with the integers 1 through 63 we used later *large*. The goal of the following problem is to compare the small cards to the first three large ones.

Problem 15.15

• *Do all the numbers from the first small card appear on the first large one? Circle the correct answer.*

Yes *No*

Why do you think this happens?

• *Do all the numbers from the second small card appear on the second large card? Circle the correct answer.*

Yes *No*

Why do you think this happens?

• *Do all the numbers from the third small card appear on the third large one? Circle the correct answer.*

Yes *No*

Why do you think this happens?

Problem 15.16

- *What is the greatest binary number one can write using two ones and one zero?*

  The number is B _____ .

- *What is the decimal value of the number?*

  The decimal value is _____ .

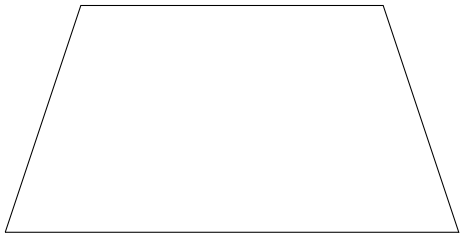
- *What is the smallest binary number one can write using two ones and one zero?*

  The number is B _____ .

- *What is the decimal value of the number?*

  The decimal value is _____ .

Problem 15.17 *Draw three straight lines to make 12 triangles.*



15.2 Homework

Use magic cards from the end of the book to show your parents the magic trick you have learned in class. Then explain how the trick works.

Problem 15.18

- *What is the greatest binary number one can write using two ones and two zeros?*

  The number is B _____ .

- *What is the decimal value of the number?*

  The decimal value is _____ .

- *What is the smallest binary number one can write using two ones and two zeros?*

  The number is B _____ .

- *What is the decimal value of the number?*

  The decimal value is _____ .

Let us make the following observation:

- One needs three cards to make a magic trick using the integers 1 through 7. There are four integers on each card.

- One needs six cards to make a magic trick using the integers 1 through 63. There are 32 integers on each card.

Problem 15.19 *Alice wants to make a similar trick with four cards.*

- *What integers would the girl use?*

She would use the integers ___ through ___ .

- *How many integers would there be on each card?*

There would be ___ integers on each card.

Problem 15.20 *Make the four cards for Alice.*

Solution: the first card will have the numbers 1, 3, 5, 7, 9, 11, 13, 15. The second card will have the numbers 2, 3, 6, 7, 10, 11, 14, 15. The third card will have the integers 4, 5, 6, 7, 12, 13, 14, 15. The fourth card will have the integers 8, 9, 10, 11, 12, 13, 14, and 15.

Nested dolls, called *matryoshki*, are popular in Russia.

Question 15.3 *What is in common between the sets of magic cards of various sizes and the matryoshki dolls?*