Electrical Circuits and Random Walks

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Adapted from notes by Padraic Bartlett and
"Random walks and electric networks" by Doyle and Snell.

This worksheet is largely about graphs. Whenever there is a graph G, $V(G)$ will denote the set of vertices of G, and $E(G)$ will denote the set of edges of G. For simplicity's sake, assume that G has no loops, that is, no vertex is connected to itself, and at most one edge connects any two vertices. For a vertex x, we will sometimes use $N(x)$ to refer to the set of neighbors of x, the vertices connected to x.

1 Random Walks

Consider a finite graph G. At one vertex, H, is home, at another vertex, B, is an all-consuming black hole. You find yourself at vertex x. If every minute you move along a randomly-selected edge from your current position to a neighboring vertex, what's the chance you successfully make it home before you stumble into the black hole?

Specifically, we define a function p from $V(G)$ to $[0,1]$: let $p(x)$ be the probability that if you start at vertex x and walk randomly, you end up at H before B.

1.1 A First Example

Consider a simple case, 4 vertices in a line, with H at one end, and B on the other. From either x or y, you pick one of the two directions to walk in, each with probability $\frac{1}{2}$, and once you have reached H or B you stay put.

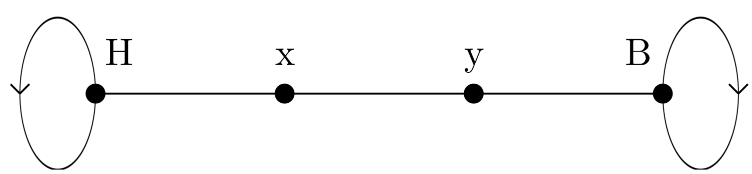


Figure 1: G, with vertices labelled.

Problem 1 Calculating the function p:

(a) What are $p(H)$ and $p(B)$?

(b) Given $p(y)$ and $p(H)$, find an expression for $p(x)$. Given $p(x)$ and $p(B)$, find an expression for $p(y)$.

(c) Use the last two parts to solve for $p(x)$ and $p(y)$.

Problem 2 In a general graph G, if the vertex x, which is neither H nor B, has neighbors $v_1, v_2, \ldots, v_n$, what expression can we find for $p(x)$ in terms of $p(v_1), p(v_2), \ldots, p(v_n)$? How can we use this to solve for p in general?

Problem 3

(a) Let G be the graph with the corners of a square as vertices, and the sides of the square as edges. If H and B are opposite corners of the square, find $p(x)$ for all vertices x.

(b) Instead of the corners and edges of a square, let G consist of the corners and edges of a 3-d cube (with H and B now opposite on the cube). Find p again.

1.2 Weighted Edges

So far, we've assumed that these random walks happen uniformly randomly, that is, whenever you choose a move randomly, each of the possible moves has the same probability. Now consider the case where you have a preference between edges to walk on, so while you still make a random choice, not each choice is equally likely. Specifically, we give each edge xy a *weight*, w_{xy}. This'll just be a positive (or at least nonnegative) real number.

We keep $p(H) = 1$ and $p(B) = 0$ as before, but for other vertices x, we change the probability of choosing an edge to be proportional to its weight. That is, if at time t you are vertex x, then at time $t + 1$, you move to a neighboring vertex y with probability

$$\frac{w_{xy}}{w_x},$$

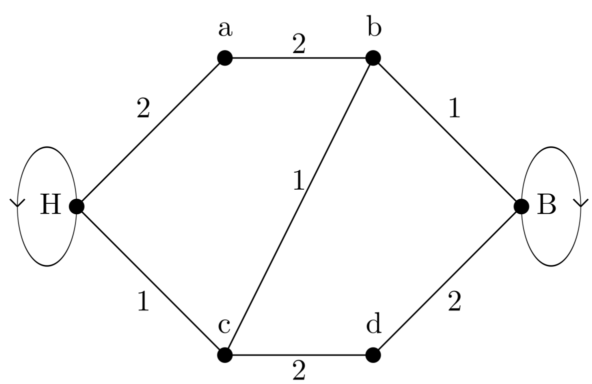
where w_x is the total of all the weights,

$$\sum_{y \text{ is a neighbor of } x} w_{xy}.$$

Problem 4 Now what is our expression for $p(x)$ in terms of the values $p(y_1), \ldots, p(y_n)$ and $w_{xy_1}, \ldots, w_{xy_n}$ where $y_1, \ldots, y_n$ are the neighbors of x?

Problem 5 Let G be a graph with 5 vertices, H, B, x, y, z. Each pair in x, y, z is connected with an edge, so these three vertices form a triangle, and H is connected only to x, while B is connected only to z. All edges have weight 1, except the edge xz, which has weight $w_{xz} = 2$. Find $p(x), p(y), p(z)$.

Problem 6 Let's try a more complicated example.



(a) What expressions can we find for $p(a), p(b), p(c)$, and $p(d)$?

(b) It is possible to solve a system of 4 linear equations in 4 variables like this straightforwardly with matrix computations, which computers can do quickly. A computer tells us that

$$p(a) = \frac{12}{19}, \qquad p(b) = \frac{5}{19}, \qquad p(d) = \frac{4}{19}.$$

Was the computer correct? If so, what is $p(c)$?

2 Circuits

And now for something completely different.

If you've worked with circuits with resistors before, or calculated things about them in a physics class, you'll be pretty familiar with this section. If not, don't worry, we will explain everything you need for this lesson.

An electrical circuit is a graph G, together with a few extra properties. We'll just be concerned with *voltage*, *current*, and *resistance*.

- Voltage is a function $V : V(G) \to \mathbb{R}$, that is, it assigns each vertex x of G a number, $V(x)$, called the voltage at x. In any circuit we think about here, we pick a vertex to call the ground, and standardize $V(\text{ground}) = 0$. We pick another vertex to be a source of electricity, and (by attaching one end of a 1 volt battery to the ground, and another end to the source) we let $V(\text{source}) = 1$.
- Current is another function $I : E(G)^{\rightarrow} \to \mathbb{R}$, where $E(G)^{\rightarrow}$ is the set of "oriented edges," that is, ordered pairs (x, y) where x and y are neighbors in the graph G. We'll use I_{xy} to describe the current of the ordered edge (x, y), and because we want to think of this as a flow of charge from x to y, the current in the opposite direction should be opposite, $I_{xy} = -I_{yx}$.
- Resistance is a function $R : E(G) \to (0, +\infty)$, giving any (unordered) edge $\{x, y\}$ a positive number. We'll use R_{xy} to describe the resistance of $\{x, y\}$.

For our purposes, these will follow only two axioms, which correspond to actual physical laws.

- Ohm's Law: The current along the edge from x to y, I_{xy}, is given by

  $$I_{xy} = \frac{V(x) - V(y)}{R_{xy}}$$

  where $V(x)$ and $V(y)$ are the voltages of these points, and R_{xy} is the resistance along the edge $\{x, y\}$, independent of direction.
- Kirchoff's Law: The sum of all currents pointing into a vertex is 0. If $x \in V(G)$, and $N(x)$ is the set of neighbors of x,

  $$\sum_{y \in N(x)} I_{xy} = 0$$

  This is because current is (basically) a flow of electrons, and the number of electrons flowing into a vertex is the same as the number flowing out, so the total change is 0.

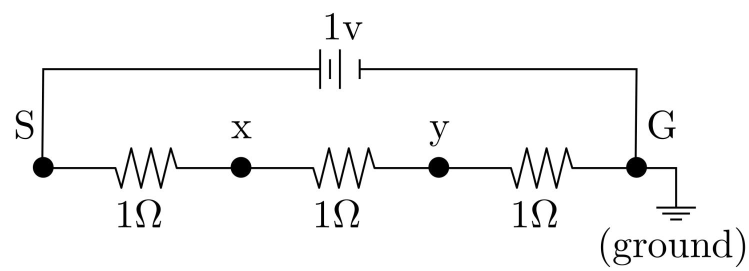
  EXCEPTION: This does not need to hold at the source or the ground vertices, where the extra current can go into or come out of the battery.

It will be convenient to also define *conductance*, which is just the reciprocal of resistance, so that the conductance C_{xy} of an edge is just $1/R_{xy}$. Then we also define (this you won't see in a physics class) the conductance $C_x = \sum_{y \in N(x)} C_{xy}$ for each vertex x to be the total conductance of the edges leading to it.

2.1 Calculating Voltage

Say we have a graph G, with a particular vertex g that we set to be the ground, with $V(g) = 0$, and a vertex s which we set to be the source, with $V(s) = 1$. We also know the resistance R_{xy} of each edge $\{x, y\}$ of G, and we will come up with a method for calculating the voltage $V(x)$ at every vertex x.

2.2 Example



This is the same graph from Figure 1, except this time as a circuit! (Note that the 1v battery is not actually an edge). We've turned it into a circuit by

- Replacing all the edges with resistors of resistance 1, measured in Ohms, hence the Ω (Capital Ohm-ega)
- Attaching a 1v battery between g and s.

Problem 7 We know by the setup of the problem that $V(g) = 0$ and $V(s) = 1$.

(a: Using Kirchoff's Law) What equations does Kirchoff's Law give us pertaining to currents out of x and currents out of y?

(b: Using Ohm's Law) Try using Ohm's Law to turn the equations about currents into equations about voltages and resistances, and find an expression for $V(x)$ in terms of other voltages, and an expression for $V(y)$ in terms of other voltages. Then solve the system of equations. Do these look familiar?

3 The Equivalence

In the last problem, you should have found that the equations for $V(x)$ and $V(y)$ were the same as the equations for $p(x)$ and $p(y)$ in the random walk problem on the same graph.

In general, we can translate from a weighted graph random walk problem to an electrical circuit problem by translating the "black hole" vertex to the ground, the "home" vertex to the source, and the weights w_{xy} to conductances C_{xy}. Assume we have a G with two selected vertices a, b. We will show that if $w_{xy} = C_{xy}$, the probability function p solving the random walk problem with home a and black hole b equals the voltage function v of the circuit with source a and ground b.

3.1 Proving p and V satisfy the same equations

Problem 8 We saw earlier that the probability function p satisfies $p(x) = \sum_{y \in N(x)} p(y) \frac{w_{xy}}{w_x}$ for every vertex x other than a and b. Show that the voltage function v satisfies

$$V(x) = \sum_{y \in N(x)} V(y) \frac{C_{xy}}{C_x},$$

using Ohm's Law and Kirchoff's Law.

3.2 Proving $p = V$

Thus if $w_{xy} = C_{xy}$, then the random walk probability function p and the voltage function V satisfy the same linear equations. To show that $p = V$, it suffices to show that there can only be one solution to these equations, which we will do in the following exercises.

Problem 9 Let q be a solution to the equations

$$q(x) = \sum_{y \in N(x)} q(y) \frac{w_{xy}}{w_x}$$

for $x \neq a, b$. Show that the maximum and minimum values of q are $q(a)$ and $q(b)$ (in some order).

Problem 10 If p, q are functions that solve these equations as well as $p(a) = q(a) = 1$ and $p(b) = q(b) = 0$, show that the function $p - q$ satisfies these equations, and $p(x) - q(x) = 0$ for every vertex x, so $p = q$.