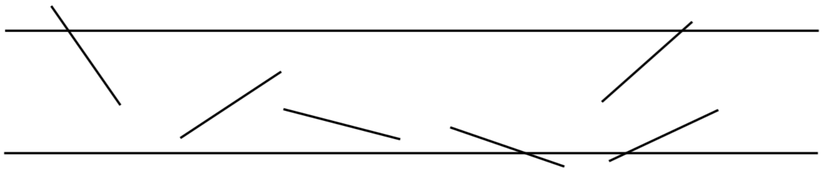
π Problems and Probability
LA Math Circle

§1 Buffon's Needle Problem

In 1777, the French noble-man Le Comte de Buffon asked the following question:

> "Suppose that you drop a short needle on ruled paper – what is then the probability that the needle comes to lie in a position where it crosses one of the lines?"

The answer will depend on the the length of the needle ℓ, and the distance between the ruled lines a distance d apart. We will assume $\ell \leq d$.



Try it yourself and see what you get!

Problem 1.1. (a) Run a Buffon experiment with at least 5 trials. Record the number of needles (toothpicks) thrown versus the number of crosses.

(b) Approximate from your data the probability P that the needle crosses a line.

(c) Estimate the value of $2/P$. Any guesses for what it should be?

Trial	# Toothpicks	#Crossings
1		
2		
3		
4		
5		
Total		

Theorem 1.2 (Buffon)

The probability that a needle of length ℓ crosses ruled lines spaced at distance d with $\ell \leq d$ is given by

$$P = \frac{2\ell}{\pi d}$$

To prove this, we will look at the *expected value* of the number of crosses. If we drop a needle of length ℓ, the expectation of the number of crossings is defined by

$$E(\ell) = P_1 + 2P_2 + 3P_3 + \dots$$

where P_1 is the probability that the needle crosses a line once, P_2 is the probability that the needle crosses at two points, etc. When $\ell \leq d$, the probability of crossing more than once is 0 so $E(\ell) = P_1$ is the probability we're looking for.

Now we can use the *linearity of expectation*. It states that whenever we have two random events, the expectation of the sum is the sum of the expectations (even if they are dependent!). In our case, linearity of expectation says that $E(x + y) = E(x) + E(y)$. In fact, linearity of expectation tells us the counter-intuitive fact that the expected number of crosses only depends on the length of the needle, not the shape! Try this out experimentally:

Problem 1.3. Break the toothpicks into halves, and do the trials as in Problem 1.1. Does linearity of expectation hold (approximately)?

Trial	# Toothpicks	#Crossings
1		
2		
3		
4		
5		
Total		

Trial	# Paperclips	# Single X'ings	# Double X'ings	# Triple X'ings
1				
2				
3				
4				
5				
Total				

Problem 1.4. If you're still not convinced of linearity of expectation, take some paper-clips and bend them any which way, so they lie flat. Do the same experiment (make sure to count multiple crossings). Does linearity of expectation hold?

Now we can deduce the form of the expectation function.

Problem 1.5. (a) Show that $E(nx) = nE(x)$ for any positive integer n.

(b) Show that $E(rx) = rE(x)$ for any positive rational number r.

(c) Show that $E(r) = cr$ for all positive rational r and some constant c.

(d) Conclude that $E(x) = cx$ for all positive real numbers x.

How can we find the value of c? The key idea is to use needles of different shapes.

Problem 1.6. Let's use "circular needles" with radius $2d$. What is the expected number of crossings?

Let I_n be the regular inscribed n-gon and C_n be the circumscribed regular n-gon of the radius $d/2$ circle.

Problem 1.7. (a) Denote the total length of I_n and C_n by $|I_n|$ and $|C_n|$ respectively. Show that

$$c|I_n| \leq 2 \leq c|C_n|$$

for all $n \geq 3$ where c is the constant from Problem 1.4.

(b) Show that

$$c = \frac{2}{\pi d}.$$

§2 Circles on a Sphere

[1] In the plane, we can define π as one half the ratio

$$\frac{\text{Circumference}}{\text{Radius}}.$$

Remarkably, the ratio is independent of the circle we use! This is related to the fact that a plane is flat (i.e., it does not curve or bend). Let's look at a non-flat surface (such as a sphere) and see what happens to the ratio above when we change the radius of a

[1] This section is based on Olga Radko's Handout "The Number π"

circle lying on it. Note that to measure the radius on a sphere, we measure it as if we are standing on the sphere and constrained to moving on the sphere.

Fix a sphere of radius R and let N be the North pole and let O be the center of the sphere. Draw a circle in the northern hemisphere on the sphere centered at N so that the angle between the vertical line ON and the line connecting O with a point P on the circle is θ. The true center (in 3 dim space) of the circle is a point O' on the line ON such that $ON \perp PO'$ (see Figure 1).

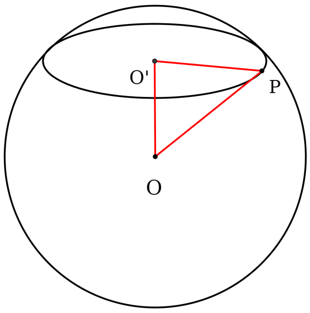


Figure 1: A circle on the sphere of radius R.

Let r be the spherical radius of the circle (it's the arc length of $\overset{\frown}{NP}$) and let $r' = |PO'|$ be the true radius.

Problem 2.1. (a) Use the right triangle $\triangle OO'P$ to express the circumference of the circle in terms of R and θ.

(b) Express θ in terms of R and r.

(c) Calculate the ratio

$$\frac{\text{Circumference}}{r} \tag{1}$$

in terms of r.

(d) Plot the ratio (1) as a function of r on the axes below. Explain your findings.

Problem 2.2. (a) What happens to the ratio when $r \to 0$? (Use the small angle approximation $\sin\theta \approx \theta$).

(b) Is there a circle on the sphere for which the ratio is exactly 2π? Why or why not?

(c) What is the smallest value of the ratio you can get, and for which circles?

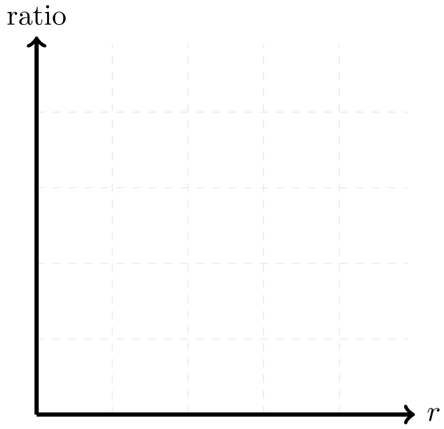


Figure 2: Spherical case

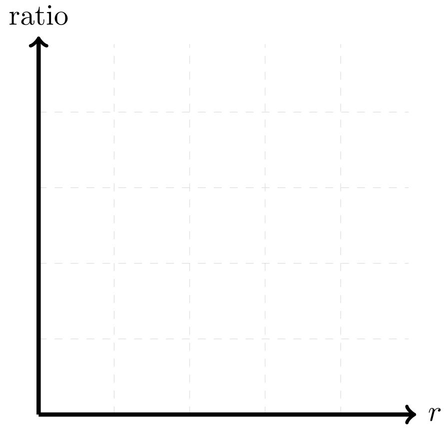


Figure 3: Hyperbolic case

§2.1 Bonus: Hyperbolic Circles

Recall the Poincaré Disc model of hyperbolic geometry. If we draw a circle of Euclidean radius R centered at the origin, we found in a previous worksheet that its hyperbolic radius r (i.e. measured in the hyperbolic distance) is related to R by

$$R = \frac{e^r - 1}{e^r + 1}.$$

It's beyond our scope to prove now, but the hyperbolic circumference is given by $\pi(e^r - e^{-r})$.

Problem 2.3. (a) Compute the ratio

$$\frac{\text{Circumference}}{\text{radius}}$$

where both the circumference and radius are measured in the hyperbolic distance. Write it in terms of $\sinh r$. (Recall that $\sinh x = (e^x - e^{-x})/2$).

(b) Plot the ratio as a function of r on the axes above. What happens as $r \to 0$?

(c) Is there any circle for which the ratio is 2π? What is the smallest you can get the ratio? Compare this to the spherical case.

§3 Co-prime Numbers

Now let's look at another probability question:

> What is the probability P that two randomly chosen positive integers are co-prime (that is, their greatest common divisor is 1)?

To interpret the question precisely, we should ask: what is the probability P_n that two numbers chosen uniformly at random from $\{1, 2, \ldots, n\}$ are co-prime? Then we can find the limiting probability as n gets larger $P = \lim_{n\to\infty} P_n$.

Problem 3.1. (a) What is the probability that two randomly selected integers q_1 and q_2 chosen from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ are coprime?

(b) In general, choose q_1 and q_2 randomly from $\{1, 2, \ldots n\}$ and let p be an prime number. What is the probability that p does not divide q_1 and does not divide q_2?

(c) Suppose $p_1, p_2, \ldots, p_k$ are all the primes less than or equal to n. Show that the probability of q_1 and q_2 being coprime is

$$P_n = \prod_{i=1}^{k} \left(1 - \frac{1}{p_i^2}\right) = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right) \cdots \left(1 - \frac{1}{p_k^2}\right).$$

(The big Pi means "product" like a big Sigma means "sum")

Now we look at the reciprocal

$$\frac{1}{P_n} = \prod_{i=1}^{k} \frac{1}{1 - \dfrac{1}{p_i^2}}.$$

Finding a closed form for this product as $n \to \infty$ was a problem first solved by Euler in 1735.

Problem 3.2. Use the geometric series formula ($\sum_{i=0}^{\infty} x^n = \frac{1}{1-x}$) to show that when $n \to \infty$ we get

$$\lim_{n\to\infty} \frac{1}{P_n} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

To evaluate this sum, we'll use Euler's idea. As you may know from calculus, we can write $\sin x$ as an "infinite polynomial" called a Taylor series.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \tag{2}$$

We will attempt to "factor" this series based on its roots.

Problem 3.3. (a) Using (2), find a series expansion of $\sin x/x$.

(b) Let's take a leap of faith and suppose we can write $\sin x/x$ in terms of its roots as you do with normal polynomials [2]:

$$\frac{\sin x}{x} = \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 + \frac{x}{3\pi}\right)\cdots \tag{3}$$

Compare the series (3) and the series from part (a). Equate the coefficients of x^2 to show that

$$-\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \dots\right) = -\frac{1}{3!}$$

(c) Conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

and hence $P = 6/\pi^2 \approx 0.6079$!

You can try this approximation method by generating random integers (e.g. `www.random.org/integers`) and seeing if they are coprime. But the convergence is generally slow, so you may want to write a program to do it!

Problem 3.4. The series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is called the *zeta function*. We calculated $\zeta(2)$. Use the same techniques to find the probability that 3 randomly selected positive integers are co-prime in terms of the zeta function.

[2]this can be justified with complex analysis

§4 Continued Fraction Approximation

Since our experimental approximation of π was somewhat inaccurate, let's look at better way to approximate real numbers using *continued fractions*.

A continued fraction is a number of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_{n-1} + \frac{1}{a_n}}}}}$$

where $a_0, a_1, a_2, \ldots a_n$ are integers. To clean up notation, abbreviate this continued fraction as $[a_o; a_1, a_2, \ldots, a_n]$. For example, $[1; 2, 3, 4] = 43/30$. To find the continued fraction of a number x, follow these steps:

1. Find the integer part of x, denoted $[x]$ (e.g. $[\pi] = 3$).
2. Find the remainder r, so
$$x = [x] + r = [x] + \frac{1}{\frac{1}{r}}$$
3. Repeat step 1 with $1/r$ instead of x.

For any rational number, this gives an essentially unique continued fraction. (Trivially, $[a_0; a_1, \ldots, a_n] = [a_0; a_1, \ldots, a_n - 1, 1]$, but it's not a big deal).

Problem 4.1. (a) What fraction is $[3; 7]$? How about $[3; 7, 15]$?

(b) Find the continued fraction for $17/12$.

(c) Find the continued fraction of $355/113$.

Problem 4.2. Prove that an irrational number has an infinite continued fraction representation $[a_0; a_1, a_2, \ldots]$

It turns out that the converse is also true: infinite continued fractions are exactly the irrational ones!

Problem 4.3. (a) Find the continued fraction of $\sqrt{2}$

(b) What is the continued fraction of the golden ratio

$$\varphi = \frac{1 + \sqrt{5}}{2}?$$

(Recall φ is a root of $x^2 - x - 1 = 0$).

(c) Find a closed form for the number $[3; \bar{3}] = [3; 3, 3, 3, 3, 3, \ldots]$.

Given a continued fraction (possibly infinite), we call the rational number obtained by truncating the continued fraction a *convergent*. E.g. $[1; 2]$ is the first convergent of $\sqrt{2}$, $[1; 2, 2]$ is the second convergent, etc. These convergents are remarkably good at approximating the original number. A theorem by Lagrange that we won't prove here asserts that they are in fact the *most accurate* out of any rational approximations with the same number of digits in the denominator.

Problem 4.4. $\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, \ldots]$. What is the most accurate rational approximation of π with no more than 5 digits in the denominator?

Problem 4.5. The Earth makes a full orbit around the sun every 365.24219878 days. The Julian calendar has 365 day years and adds 1 day every 4 years for an average of 365.25 days in a year. This is too long by about 1 day every century.

(a) The Gregorian calendar (our calendar) adds 1 day every 4 years, subtracts 1 day every 100 years, and adds 1 day every 400 years. What is the error in the average number of days per year?

(b) What is the fourth convergent of 0.24219878? (You can use a calculator).

(c) Propose a simple new calendar based on part (b) and show its error is less than that of the Gregorian calendar.

§5 Flipping Coins

Let's look at one more probability question.

> Flip a fair coin $2n$ times. Suppose P_n is the probability that you get the same number of heads as tails. What happens to the probability as n gets larger ($n \to \infty$)?

Problem 5.1. Determine the probability P_n that there are equal number of heads and tails after flipping the coin $2n$ times.

The interesting part is what happens as $n \to \infty$. For this, we'll use another amazing representation of π:

Problem 5.2. (a) Use Euler's infinite product representation (3) to show that

$$\frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots = \prod_{n=1}^{\infty} \left(\frac{2n}{2n-1} \cdot \frac{2n}{2n+1} \right) = \frac{\pi}{2}.$$

This is known as the Wallis product formula.

(b) Show that the Wallis product formula is equivalent to

$$\lim_{n\to\infty} \frac{2^{4n}(n!)^4}{((2n)!)^2(2n+1)} = \frac{\pi}{2} \tag{4}$$

Now let's do some *asymptotic analysis*. This means we'll only be concerned about what happens as $n \to \infty$. For two functions $f(n)$ and $g(n)$, we say $f \sim g$ or "f is asymptotic to g" if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1.$$

Compare this to the Big-O notation $f = O(g)$ we used for algorithms.

Problem 5.3. Give an example of two functions $f(n)$ and $g(n)$ such that $f = O(g)$ but f is not asympotic to g.

Now we can answer our original question about the aymptotic behavior of P_n as n goes to infinity.

Problem 5.4. (a) Let

$$f(n) = \frac{2^{2n}(n!)^2}{(2n)!\sqrt{2n+1}}$$

(it's the square root of the left side of (4)). What is $f(n)/P_n$?
(Note that $\sqrt{2n+1} \sim \sqrt{2n}$.)

(b) What function is P_n asympototic to?

(c) Check your answer using Stirling's approximation, which says

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$$