The number π

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One Man's Definition Of Pi (From Godling's Glossary, By Dave Krieger):
π

1. The Greek letter P or p, corresponding to the roman p.

2. A number, represented by said letter, expressing the ratio of the circumference of a perfect circle to its diameter. The value of π has been calculated to many millions of decimal places, to no readily apparent purpose: no perfect circles or spheres exist in nature, since matter is composed of atoms and therefore lumpy, not smooth. Nature herself sometimes takes to rounding off the more extreme decimals of numbers when they get sufficiently small, as Prof. Heisenberg has pointed out. However, the continued extension of π provides a harmless exercise of computer power which would otherwise be misused playing Quake or surfing pointless web sites.

1 Circumference and Area of Polygons

Experiment:

1. For each of the regular polygons, calculate the perimeter as a function of the radius r. Put the data into the table below:

Number of sides	P	A
4		
6		
8		

2. For each of the polygons, use the provided graphing paper to plot perimeter P as a function of the radius r. Is it a linear function? Why? What is the slope of the graph?

3. Do the same as Problem 2 with area against the square of the radius r^2.

4. Why can you use Problems 2 and 3 to approximate π? Use your calcuations for 8 sides to estimate the value of π in two different ways.

5. Show geometrically that these two constructions of π in Problem 4 are the same. In other words, given a circle has circumference $2\pi r$, show that it has area πr^2. (Hint: Cut up the the circle into sectors.)

2 Circles on a sphere

It is an interesting and non-trivial fact that the ratio

$$\frac{\text{Circumference}}{\text{Radius}}$$

is independent of the radius of the circle on the plane. This fact requires a proof which can be given in a geometry course using some ideas from calculus (taking limits). On a very fundamental level, it is related to the fact that a plane is *flat* (i.e., it does not curve or bend).

We will take a look at a non-flat surface (such as a sphere) and see what happens to the ratio above when we change the radius of a circle lying on a sphere.

First, we need to understand what to call a *radius* for a circle lying on a sphere. Assume that you put the ruler's fixed end at the "north pole" of your sphere and draw a circle. Forget that the surface of the sphere lies in our three dimensional space. (E.g., imagine that you are a 2-dim bug living on the sphere. How would you measure the radius of your circle? You would stay on the surface and call the length of the shortest possible path connecting the center with a point on the circle it's radius).

You can use an experimental approach, or some trigonometry (section 2.1), or both.

2.1 Circles on a sphere: trigonometric approach

- Let N be the "North pole" of your sphere and O be the center of the sphere.
- Consider a horizontal circle in the "northern hemisphere" so that the angle between the vertical line ON and the line connecting O with a point P on the circle is θ;
- The true center (in 3 dim space) of the circle is a point O' on the line ON so that $ON \perp PO'$.
- Let R be the radius of the sphere;
- Let r be the "radius" of the circle as measure on the sphere (r equals to the arc length of the arc NP); The largest "radius" is that of the equator. It equals to the length of a quarter of the largest circle, i.e., $\frac{2\pi R}{4} = \frac{\pi R}{2}$.
- Let r' be the true radius of the circle, $r' = |O'P|$.

Compute the ratio $\frac{\text{Circumference}}{\text{"Radius"}}$ as follows:

1. Consider the right angle triangle $\triangle OO'P$ and express the true radius of the circle, r', via R and θ:
$$r' =$$
2. The circumference of the circle is equal to
$$\text{Circumference} = 2\pi r' =$$
3. The radius of the circle as measured along the sphere is the arc length of the arc NP subtended by angle θ of a circle of radius R. Thus,
$$r =$$
4. The last equation allows us to express θ in terms of R and r:
$$\theta =$$
5. Substituting θ into the expression for the circumference, we get
$$\text{Circumference} =$$
6. Finally, the ratio
$$\frac{\text{Circumference}}{\text{"Radius"}} =$$
7. Plot Circumference/"Radius" against r on the graph paper provided. Fix R the radius of the sphere. Plot Circumference against radius r on the graph paper provided. Explain the findings.

Now use this expression for the ratio to answer the following questions:

1. We will prove geometrically that $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$. Draw the first quadrant of the unit circle with origin O and label (1,0) as A. Define B to be another point on the perimeter of the circle at an angle θ from the segement OA. Extend the line segment OB to a point C so that ΔOAC is a right triangle. Use this picture to prove $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$.
2. What happens to the ratio when the radius of the circle r is small compared to the radius of the sphere R? (By Problem 1, you may use the approximation $\frac{\sin\theta}{\theta} \approx 1$ for small values of θ.)

3. Compute the ratio for given values of $\frac{r}{R}$ and fill in the table below:

$\frac{r}{R}$	$\frac{\text{Circumference}}{\text{Radius}}$
$\frac{\pi}{6}$	
$\frac{\pi}{4}$	
$\frac{\pi}{3}$	
$\frac{\pi}{2}$	

4. Is there a circle on a sphere for which the ratio is exactly 2π? Why or why not?

5. What is the smallest value of the ratio that you can get? What are the circles that produces this ratio?

3 Buffon's needle

Let's take a look at the simplest version of the following experiment, called the *Buffon's needle*, which was proposed by Georges-Louis Leclerc, Comte de Buffon in 1733:

Suppose that we have a floor made of parallel strips of wood and we drop a needle onto the floor. What is the probability that the needle will lie across the line between the two strips of wood?

The answer to this probability question not only involves the number π, but also provides a method of computing π experimentally!

We will deal with the simplest case when the width of strips of wood equals to the length of the needle.

Assume that the length of the needle and the distance between the nearest lines are both equal to 1. Let d be the distance from the needle's center to the nearest line. Let θ be the angle between the needle and the lines.

1. Determine all possible values of d and θ:

$$\leq d \leq$$
$$\leq \theta \leq$$

2. Let O be the center of the needle and P be the point of intersection of the needle (or its extension) with the nearest line. Express $l = OP$ in terms of d and θ.

3. What are conditions on d and θ so that the needle crosses a line?

4. On the graph paper provided, put θ as the variable on the horizontal axis and d as the variable on the vertical axis.

   (a) Mark the rectangle corresponding to the possible values of θ and d.

   (b) Shade the region that corresponds to the values of θ and d such that the needle crosses a line.

   (c) The probability that the needle crosses a line equals to

$$P = \frac{\text{Area (shaded region)}}{\text{Area (rectangle)}};$$

   Estimate this probability by finding the area of the rectangle and estimating the area of the shaded region using the graph paper:

squares inside shaded region	squares inside the rectangle	probability	$\frac{2}{\text{probability}}$

5. (Challenge) If you are familiar with integral calculus, take a look at the following computation of this probabiliy. Otherwise, take a look at the answer:‘

$$P = \frac{\text{Area (shaded region)}}{\text{Area (rectangle)}} = \frac{\int_0^\pi \frac{1}{2}\sin\theta d\theta}{\frac{1}{2}\cdot\pi} = \frac{1}{\pi}\int_0^\pi \sin\theta d\theta = \frac{1}{\pi}\cos\theta|_0^\pi = \frac{2}{\pi}.$$

Using a calculator, compute the value $\frac{2}{\pi}$ and compare it with the probability you have found by estimating the areas under curves.

6. This relation between the probability in Buffon's needle experiment and π led to many experimental estimates of π. Several enthusiasts made the experiment of dropping the needle many times and recording the data. The probability was computed as
$$P = \frac{\#(\text{tossings such that needle crossed a line})}{\#(\text{all tossings})}$$
and then π was estimated as $\pi \approx \frac{2}{P}$. There are many computer implementations of this (search for the Buffon's needle experiment if you would like to find one!)