# Comments about the Hard Problems

**Problem 1)** How many zeroes are at the end of 100!?

**Comment 0.1** *It suffices to count powers of* 5 *(reinforce why).*

**Problem 2)** The equation $x^2 + y^2 + z^2 = 10xyz$ has no integral solutions, except $x = y = z = 0$. Prove it.

**Comment 0.2** *This is an infinite descent (a tricky concept) problem, using either* 2 *or* 5 *(i.e. the divisors of* 10*).*

**Problem 3)** Given three integers, $\{a, b, c\}$, prove that we can find two of them – say, $a$ and $b$ – such that $10 \mid a^3b - ab^3$.

**Comment 0.3** *In light of the previous problems, they should be able to figure out that we need to prove*

$$2, 5 \mid a^3b - ab^3.$$

*Expressing* $a^3b - ab^3 = ab(a + b)(a - b)$ *takes care of division by 2. For division by 5, by the pigeon-hole principle, at least two of* $\{a, b, c\}$ *must belong to either*

$$\{1, 4\} \text{ or } \{2, 3\}.$$

For the subsequent problems, I'd give out the right numbers to "mod out by" as hints.

**Problem 4)** Define the sequence $\{a_n\}$ by setting $a_1 = 2, a_2 = 5$, and

$$a_{n+1} = (2 - n^2)a_n + (2 + n^2)a_{n-1},\ n \geq 2.$$

Do there exist indices, $p, q, r$ such that $a_p \times a_q = a_r$?

**Comment 0.4** *Working mod 3, every* $a_i \equiv 2$*, so the answer is no.*

**Problem 5)** $2^{29}$ is a nine-digit number, with no repeated digits. Which digit does not appear?

**Comment 0.5** *Working mod 9, we note that* $2^8 \equiv 1$*. Then:* $2^{29} = (2^8)^3(2^5) \equiv -4 \mod 9$*, so we're missing the digit* 4*.*

**Problem 6)** The last four digits of a perfect square are equal. Prove that they are equal to zero.

**Comment 0.6** *This is a pain to write out. Get them started by noticing that the choices for such a final digit, a, are* $a \in \{0, 1, 4, 5, 6, 9\}$*. Rule out the positive a's by expressing your number as* $1111a \mod 10000$ *and work mod* 8 *and mod* 16*.*

**Problem 7)** Given a prime $p$, and two integers, $a, b$, working mod $p$, express

$$(a+b)^p$$

in terms of $a$ and $b$. (This result is coolest if Fermat's Little Theorem is taken into account.)

**Comment 0.7** *Encourage them to try this one.*

**Problem 8)** Earlier, we saw that for every prime $p$, $(p-1)! \not\equiv 0 \mod p$. Compute $1! \mod 2$, $2! \mod 3$, $4! \mod 5$, and $6! \mod 7$. Is there a pattern that emerges? Can you prove it?

**Comment 0.8** *This is Wilson's Theorem. The key is to group terms with their mod $p$-inverses*