Sequences II

Matthew Gherman and Adam Lott

February 10, 2019

Subsequences

Definition 1. Let (s_n) be a sequence. A *subsequence* is any sequence of the form $t_k = s_{n_k}$ where $n_1 < n_2 < n_3 < \dots$ is an increasing sequence of natural numbers. We say a number L is a *subsequential limit* of (s_n) if there exists a subsequence of (s_n) converging to L.

Exercise 1.
(a) Let $s_n = (-1)^n$. Show that 1 and -1 are subsequential limits. Are there any others?

(b) Let $s_n = n^{(-1)^n - 1}$. Find all of the subsequential limits of (s_n).

Exercise 2. Show that L is a subsequential limit of (s_n) if and only if the following holds:
For all $\epsilon > 0$ and all $M > 0$ there exists $n > M$ such that $|s_n - L| < \epsilon$.

Exercise 3 (CHALLENGE)**.** Let (s_n) be a sequence and let E be the set of all subsequential limits of (s_n). Recall from last week the definitions

$$\limsup_{n\to\infty} s_n = \lim_{n\to\infty}\left(\sup_{m>n} s_m\right) \qquad \liminf_{n\to\infty} s_n = \lim_{n\to\infty}\left(\inf_{m>n} s_m\right).$$

(a) Prove that $\limsup_{n\to\infty} s_n = \sup E$ and $\liminf_{n\to\infty} s_n = \inf E$.

(b) Prove that there exists a subsequence converging to $\sup E$ and another (possibly different) subsequence converging to $\inf E$.

Exercise 4. Prove that a sequence (s_n) converges to L if and only if every subsequence has a further sub-subsequence converging to L.

Exercise 5 (CHALLENGE)**.** Let $(s_n) \subseteq [0,1]$ be a sequence. Prove that (s_n) has a convergent subsequence. This result is often called the *Bolzano-Weierstrass theorem.*

Cauchy sequences

Definition 2. A sequence (s_n) is called a *Cauchy sequence* if for all $\epsilon > 0$, there exists an N such that $|s_n - s_m| < \epsilon$ for all $n, m \geq N$.

Exercise 6. Prove that if a Cauchy sequence has a subsequence converging to L, then the whole sequence converges to L.

Exercise 7.
(a) Prove that if (s_n) converges, then it is a Cauchy sequence.

(b) (CHALLENGE) Prove that if (s_n) is a Cauchy sequence, then it converges.

Exercise 8. We just saw that a sequence converges if and only if it is a Cauchy sequence. Why bother making a second definition if they are equivalent? Think about the differences between the two definitions and why each might be better in different situations.

Exercise 9. Suppose (s_n) and (t_n) are Cauchy sequences. Prove that (u_n) converges, where u_n is defined as $|s_n - t_n|$. (Hint: triangle inequality)

Miscellaneous

Definition 3. A set $E \subseteq \mathbb{R}$ is called *closed* if for any sequence (s_n) with $s_n \in E$ for all n, if (s_n) converges to s then $s \in E$ also.

Exercise 10. Prove that the interval $[0, 1]$ is a closed set.

Exercise 11. Suppose (s_n) is a bounded sequence. Define

$$A = \{a \in \mathbb{R} : s_n < a \text{ for only finitely many } n\} \qquad B = \{b \in \mathbb{R} : s_n > b \text{ for only finitely many } n\}.$$

Prove that $\limsup_{n\to\infty} s_n = \inf B$ and $\liminf_{n\to\infty} s_n = \sup A$.

Exercise 12. Let (s_n) be a sequence and suppose that $\lim_{n\to\infty} \left|\frac{s_{n+1}}{s_n}\right| = L$.
(a) If $L < 1$, prove that $\lim\limits_{n\to\infty} s_n = 0$.
(b) If $L > 1$, prove that $\lim\limits_{n\to\infty} s_n = \infty$.
(c) If $L = 1$, give four different examples to show that s_n can converge to 0, converge to a nonzero limit, diverge to ∞, or oscillate.