Continued Fractions II

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1 Introduction

Last week, we introduced continued fractions and learned how to calculate them. This week, we will investigate an important application.
Recall an infinite continued fraction is an expression of the form

$$[a_0, a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ldots}}}}$$

where $a_0, a_1, a_2, \ldots$ are natural numbers.

Definition 1. Let $\alpha = [a_0, a_1, a_2, \ldots]$ be an infinite continued fraction (aka an irrational number). The *nth convergent to* α is the rational number $[a_0, a_1, \ldots, a_n]$ and is denoted $C_n(\alpha)$.

Exercise 1. Calculate the following convergents and write them in lowest terms:
(a) $C_3([1, 2, 3, 4, \ldots])$ (b) $C_4([0, \overline{2, 3}])$ (c) $C_5([\overline{1, 5}])$

Exercise 2. Recall from last week that $\sqrt{5} = [2, \overline{4}]$. Calculate the first five convergents to $\sqrt{5}$ and write them in lowest terms. Do you notice any patterns? (Hint: look at the numbers $\sqrt{5} - C_j(\sqrt{5})$ for $1 \leq j \leq 5$)

2 Properties of Convergents

In this section, we want to show that the nth convergent to a real number α is the best approximation of α with the given denominator. Let $\alpha = [a_0, a_1, \ldots]$ be fixed, and we will write C_n instead of $C_n(\alpha)$ for short. Let p_n/q_n be the expression of C_n as a rational number in lowest terms. We will eventually prove that $|\alpha - C_n| < \frac{1}{q_n^2}$, and there is no better rational estimate of α with denominator less than or equal to q_n.

First we want the recursive formulas $p_n = a_n p_{n-1} + p_{n-2}$ and $q_n = a_n q_{n-1} + q_{n-2}$ given $p_{-1} = 1$, $p_0 = a_0$, $q_{-1} = 0$, and $q_0 = 1$.

Exercise 3. Verify the recursive formula for $1 \leq j \leq 3$ for the convergents C_j of:
(a) $[1, 2, 3, 4, \ldots]$ (b) $[0, \overline{2, 3}]$ (c) $[\overline{1, 5}]$

Exercise 4 (CHALLENGE)**.** Prove that $p_n = a_n p_{n-1} + p_{n-2}$ and $q_n = a_n q_{n-1} + q_{n-2}$ by induction.

(a) As the base case, verify the recursive formulas for $n = 1$ and $n = 2$.

(b) Assume the recursive formulas hold for $n \leq m$ and show the formulas hold for $m + 1$.

Exercise 5. Using the recursive formula from Exercise 4, we will show that $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$.

(a) What is $p_1 q_0 - p_0 q_1$?

(b) Substitute $a_n p_{n-1} + p_{n-2}$ for p_n and $a_n q_{n-1} + q_{n-2}$ for q_n in $p_n q_{n-1} - p_{n-1} q_n$. Simplify the expression.

(c) What happens when $n = 2$? Explain why $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$.

Exercise 6 (CHALLENGE)**.** Similarly derive the formula $p_n q_{n-2} - p_{n-2} q_n = (-1)^{n-2} a_n$.

Exercise 7. Recall $C_n = p_n/q_n$. Show that $C_n - C_{n-1} = \frac{(-1)^{n-1}}{q_{n-1} q_n}$ and $C_n - C_{n-2} = \frac{(-1)^{n-2} a_n}{q_{n-2} q_n}$. (Hint: use Exercise 5 and $p_n q_{n-2} - p_{n-2} q_n = (-1)^{n-2} a_n$ respectively)

In Exercise 2, the value $\alpha - C_n$ alternated between negative and positive and $|\alpha - C_n|$ got smaller each step. Using the relations in Exercise 7, we can prove that this is always the case. Specifically, α is always between C_n and C_{n+1}.

Exercise 8. Let's figure out how well the nth convergents estimate α. We will show that $|\alpha - C_n| < \frac{1}{q_n^2}$.

(a) Note that $|C_{n+1} - C_n| = \frac{1}{q_n q_{n+1}}$.

(b) Why is $|\alpha - C_n| \leq |C_{n+1} - C_n|$?

(c) Conclude that $|\alpha - C_n| < \frac{1}{q_n^2}$.

We are now ready to prove a fundamental result in the theory of rational approximation.

Exercise 9 (Dirichlet's approximation theorem)**.** Let α be any irrational number. Prove that there are infinitely many rational numbers p/q such that $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$.

Exercise 10 (CHALLENGE)**.** Prove that if α is *rational*, then there are only *finitely* many rational numbers p/q satisfying $|\alpha - p/q| < 1/q^2$.

The above result shows that the nth convergents estimate α extremely well. Are there better estimates for α if we want small denominators? In order to answer this question, we introduce the Farey sequence.

Definition 2. The *Farey sequence* of order n is the set of rational numbers between 0 and 1 whose denominators (in lowest terms) are $\leq n$, arranged in increasing order.

Exercise 11. List the Farey sequence of order 4. Now figure out the Farey sequence of order 5 by including the relevant rational numbers in the Farey sequence of order 4.

Exercise 12. Let $\frac{a}{b}$ and $\frac{c}{d}$ be consecutive elements of the Farey sequence of order 5. What does $bc - ad$ equal?

Exercise 13 (CHALLENGE)**.** Prove that $bc - ad = 1$ for $\frac{a}{b}$ and $\frac{c}{d}$ consecutive rational numbers in Farey sequence of order n.
(a) In the plane, draw the triangle with vertices (0,0), (b, a), (d, c). Show that the area A of this triangle is $\frac{1}{2}$ using Pick's Theorem. Recall that Pick's Theorem states $A = \frac{B}{2} + I - 1$ where B is the number of lattice points on the boundary and I is the number of points in the interior. (Hint: B=3 and I=0)

(b) Show that the area of the triangle is also given by $\frac{1}{2}|ad - bc|$.

(c) Why is $bc > ad$?

(d) Conclude that $bc - ad = 1$.

Exercise 14. Use the result of Exercise 13 to show that there is no rational number between C_{n-1} and C_n with denominator less than or equal to q_n. Conclude that if a/b is any rational number with $b \leq q_n$, then $\left|\alpha - \frac{a}{b}\right| \geq \left|\alpha - \frac{p_n}{q_n}\right|$.

Remark 1. What the above exercise shows is that relative to the size of the denominator, the convergents of the continued fraction expansion of α are the absolute best rational approximations to α.

Exercise 15 (CHALLENGE)**.** Prove the following strengthening of Dirichlet's approximation theorem. If α is irrational, then there are infinitely many rational numbers p/q satisfying $\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}$.
(a) Prove that $(x+y)^2 \geq 4xy$ for any real x, y. (b) Let p_n/q_n be the nth convergent to α. Prove that

$$\left|\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}\right|^2 \geq 4\left|\frac{p_n}{q_n} - \alpha\right|\left|\frac{p_{n+1}}{q_{n+1}} - \alpha\right|$$

(Hint: α lies in between $\frac{p_n}{q_n}$ and $\frac{p_{n+1}}{q_{n+1}}$).

(c) Prove that either $\frac{p_n}{q_n}$ or $\frac{p_{n+1}}{q_{n+1}}$ satisfies the desired inequality (Hint: proof by contradiction).

(d) Conclude that there are infinitely many rational numbers p/q satisfying $\left|\alpha - \frac{p}{q}\right| < \frac{1}{2q^2}$.