Introduction to Paradoxes

LA Math Circle Beginner Group

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1 Warm Up!

1. Assume that three sailors are caught by a group of pirates and kept blindfolded on the pirate ship. Lucky for them, they still get a chance to save their lives. Every sailor is asked to choose a hat from a pile that consists of three black hats and two white hats, and they are asked to line up in the order of height with the first person being the shortest. In this case, the third person sees the hats of the first two people, and the second person sees the hat of the first person, and the first sailor sees no hat. The leader of the pirates who is infatuated with logic puzzles at the time sets a rule, which is that if any one of the three sailors states the color of his own hat correctly, all of the three sailors are safe. At the same time, sailors are only allowed to give their answer if they are 100 percent sure, and if they violate this rule, the pirate will know instantaneously and throw all of them into the freezing ocean.

   Then, can you identify the sailor(s) who can save the crew. How can he do it?

2. "What you have not lost you have. But you have not lost horns. Therefore, you have the horn." - Eubulides

   Can you point out the logical fallacy of this statement?

After the two exciting warm-up problems, let's get into today's main dish–paradoxes, but let's first see what is **not** a paradox!

2 Pseudo-Paradoxes

All of the following "paradoxes" arise from **fallacious reasoning**, so try to find what's wrong with every deduction!

I. 5=4 Paradox

The "proof" is shown below. Please identify and correct the mistake(s) of the reasoning and fill out the table.

$$-20 = -20$$
$$25 - 45 = 16 - 36$$
$$5^2 - 5 \times 9 = 4^2 - 4 \times 9$$
$$5^2 - 5 \times 9 + \frac{81}{4} = 4^2 - 4 \times 9 + \frac{81}{4}$$
$$(5 - \frac{9}{2})^2 = (4 - \frac{9}{2})^2$$
$$5 - \frac{9}{2} = 4 - \frac{9}{2}$$
$$5 = 4$$

Theorem/Property Used in the Step	When Can We Use the Theorem/Property?	Are We Allowed to Use It Here?

II. 2=1 Paradox

The reasoning is shown below. Please identify and correct the mistake(s) of the deduction and fill out the table.

$$X = Y$$
$$X^2 = XY$$
$$X^2 - Y^2 = XY - Y^2$$
$$(X + Y)(X - Y) = Y(X - Y)$$
$$X + Y = Y$$
$$2Y = Y$$
$$2 = 1$$

Theorem/Property Used in the Step	When Can We Use the Theorem/Property?	Are We Allowed to Use It Here?

III. All Horses Have the Same Color

Proof by Induction:

Before introducing the paradox, we need to first introduce **proof by induction**. Here are the steps of proof by induction:

1. Prove that the statement holds for the first term (normally 0 or 1).
2. Assume that it holds for the n^{th} term and prove that it holds also for the $n + 1^{th}$ term.

Then, let's take a look at the "reasoning" of this "Paradox"!

1. When there is only 1 horse, all horses have the same color.

2. Assume that the statement holds for N horses. Then given any $N + 1$ horses, if the first horse is excluded, you will get N horses, and by the assumption, they should all have the same color. If you exclude the last horse, you are still going to have N horses, and by assumption, they all have the same color. This tells us that the first horse, the last horse, and the rest of the horses all have the same color. Therefore, all horses have the same color.

Can you offer an explanation to this "paradox"?

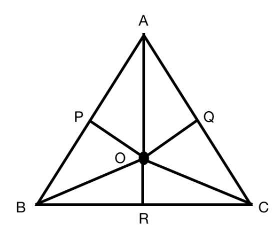
IV. The Fallacy of Isosceles Triangle (Optional)

If you finish the handout before the end of the class, you can work on this problem, but first **skips** it!

Fun Fact: This fallacy was first presented by Lewis Carroll, the author of *Alice's Wonderland*.

The "proof" is shown below. Please identify and correct the mistake(s) of the reasoning.

Let ABC be a triangle, O be the intersection of the bisector of $\angle A$ and the perpendicular bisector of BC. Let OP be perpendicular to AB and OQ be perpendicular to AC.



(1) According to the description, we get: $BR = RC, \angle ORB = \angle ORC = 90°, BO = CO$

(2) Since $\angle APO = \angle AQO = 90°$, $\angle PAO = \angle QAO$, $AO = AO$, $\triangle APO \cong \triangle AQO$. This tells us that $PO = QO$ and $AP = AQ$

(3)For $\triangle BPO$ and $\triangle CQO$, we get $\angle BPO = \angle CQO = 90°$, $BO = CO$, $OP = OQ$, so $\triangle BPO \cong \triangle CQO$. This tells us that $BP = CQ$.

(4)Then we get $AB = BP + AP = CQ + QA = AC$, meaning that $\triangle ABC$ is isoseceles!

(**HINT**: Draw your own picture!)

Let's now study some serious paradoxes!

3 Infinity-Related Paradoxes

I. The Mysterious Grandi's Series

1. Find the value of the following expression:

   (a) $1 - 1 + ... + 1 - 1$ with even numbers of 1's.

   (b) $1 - 1 + ... + 1 - 1 + 1$ with odd numbers of 1's.

2. (a) Based on your answer to part (a), can you guess the value of the infinite series $1 - 1 + 1 - 1 + 1 \cdots$?

   (b) Based on your answer to part (b), can you guess the value of the infinite series $1 - 1 + 1 - 1 + 1 \cdots$?

   (c) Do your answers to problem $2a$ and problem $2b$ match? What does this tell you about infinite series comparing to adding finite terms together ?

Are you surprised by your finding? If not, bear with me, and if yes, don't be too hyped now, because something more astonishing is going to happen!

Let's try to find the value in one more way and see what we get.

3. (a) Given that $S = 1 - 1 + 1 - 1 + 1 \cdots$, compute $1 - S$

(b) Compare S and $1 - S$. What can you conclude about the value of S?

(c) Does your answer to problem 3(b) match with your answers in problem 2(a) and 2(b)? If not, can you guess which one is right?

Mystery Revealed!

Why does the series $1 - 1 + \cdots$ behave so strangely? It is because:

1. The series is infinite, meaning it never ends.
2. It might not have be possible to find its value.

Therefore, we have to be very careful when applying the rules for finite sums to infinite series. After we have identified the problem, can you guess what is the next thing we should do to solve this problem?

As you have already had your idea, let's see what mathematicians did to solve the puzzle.

The problem remained unsolved for a century until the great French mathematician *Augustin-Louis Cauchy* (1789-1857) has offered a definition of summation for infinite series, and the definition is as follow:

$$S = \lim_{n\to\infty}(a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n)$$

The above definition might be confusing, so here is an explanation for it:

When you try to find the value of an infinite series, you need to start with finite sum. Here are the steps you need to follow:

(a) You need to find the first term, the sum of the first 2 terms, the sum of the first 3 terms, etc.

(b) You need to find a finite value the sum is approaching as more and more terms are added. The value is called the limit of the series, which is S in the definition.

(c) If the value exists, then the value is the sum of the infinite series. If the value does not exist, then the infinite series has **no sum** under this definition.

Let's first practice what we just learned and then apply it to the mysterious Grandi's series $(1 - 1 + 1 - 1 + 1 \cdots)$!

Important Notation: Let $S_1 = a_1$(the first term), $S_2 = a_1 + a_2$(the sum of the first two terms), etc. $S_n = a_1 + a_2 + \cdots + a_{n-1} + a_n$ (the sum of the first n terms)

4. For the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$, complete the table.

S_1	S_2	S_3	S_4	S_5	S_6	$\cdots$	S_n	$\cdots$
						$\cdots$		$\cdots$

What does your answer tell you about the value of the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$?

After examining this example, let's see what we can do with our Grandi's series! (Remember that Grandi's series is: $1 - 1 + 1 - 1 + \cdots$)

5. Please complete the table below.

S_1	S_2	S_3	S_4	S_5	S_6	$\cdots$	S_n	$\cdots$
						$\cdots$		$\cdots$

6. What happens when we take more and more terms? Does the sum approach any fixed value?

7. What does you answer to problem 6 tell you about the value of the series $1 - 1 + 1 - 1 + \cdots$ according to Cauchy's definition?

8. Compare your answer to problem 4 and problem 7 and offer an explanation to the difference.

Besides the classic definition of summation, there are also other alternatives, and *Cesàro summation* is an example. Let $S_n = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$, then the Cesàro sum for an infinite series is:

$$S^{'} = \lim_{n\to\infty} \frac{1}{n}(S_1 + S_2 + S_3 + \cdots + S_{n-1} + S_n)$$

In other words, according to Cesàro's method, we try to find the number that the average of sums, $\frac{1}{n}(S_1+S_2+S_3+\cdots+S_{n-1}+S_n)$, is approaching as n gets larger.

9. First, let's apply the idea to the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$! Based on the table you made in problem 4, can you complete the following table?

S_1	$\frac{1}{2}(S_1 + S_2)$	$\frac{1}{3}(S_1 + S_2 + S_3)$	$\cdots$	$\frac{1}{n}(S_1 + S_2 + \cdots + S_n)$	$\cdots$
			$\cdots$		$\cdots$

According to the table above, what can you conclude about the Cesàro sum of the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$? Does it match with the Cauchy sum that you derived for problem 4?

However, are the Cauchy sum and Cesàro sum of a series always equal? Let's now examine our good friend, Grandi's series!

10. Based on the Cesàro's method and the table you made for problem 5, can you complete the following table for Grandi's series?

S_1	$\frac{1}{2}(S_1 + S_2)$	$\frac{1}{3}(S_1 + S_2 + S_3)$	$\cdots$	$\frac{1}{n}(S_1 + S_2 + \cdots + S_n)$	$\cdots$
			$\cdots$		$\cdots$

11. Based on the table you just made, what can you conclude about the sum of the Grandi's series under Cesàro summation? Does your answer match with any of the answers to problem 2 and problem 3?

12. Can you guess the relation between the Cauchy sum and the Cesàro sum of an infinite series?

After understanding how summation works for infinite series, let's together study a paradox that is related to it!

II. Zeno Paradox

"In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead" - *Physics*, Aristotle

Consider the following scenario, Achilles ($10m/s$) and the tortoise ($5m/s$) have a running competition, and due to the huge difference between their speeds, Achilles allows the tortoise to have a head start of 20m. Then, the tortoise claims that Achilles can never overtake him, and his reasoning is as follow:

In order for Achilles to overtake him, Achilles should at least reach the tortoise's starting point, but when the great runner gets there, the tortoise has advanced further. The next time when Achilles reaches the point where the tortoise has previously resided, the tortoise has advanced further. Therefore, Achilles can never overtake the tortoise in finite time.

Is the reasoning valid? Let's think it through.

Let's assume that we decide to record the competition and play the video on our computer, and we pause if and only if Achilles has reached the tortoise's starting point of the same interval. Let n be the number of times we press "stop," and the interval between the nth stop and $(n-1)$th stop is known as the nth interval

1. How many intervals will there be if the entire video ends as soon as Achilles gets to the tortoise?

2. What is the distance traveled by Achilles during the n^{th} interval? (Use the numbers provided in the question)

   (Hint: It might be helpful if you visualize what is going on here)

3. How long does the Achilles need to spend to complete the n^{th} interval?

4. Can you write the expression of the total time of the video?

5. Can you compute the total time based on the Cauchy's definition of sum?

6. Does your answer in part 5 support or refute Aristotle's claim?

7. If your answer in part 6 is "refute," can you identify the mistake in the reasoning?

After the introduction of infinity-related paradoxes, let's take a look at another significant category of paradoxes! Believe it or not, it is also going to blow your mind!

4 Logic Paradox

I. The Heap Paradox

Assume that we have a heap of sand, and we now remove one grain of sand from the heap at a time. We assume that we cannot turn a heap into non-heap by removing a single grain, and it is clear that one grain does not make up a heap. Then when does the heap turn into non-heap? If you find the problem to be paradoxical, can you come up with a method to solve it?

Let's now study and evaluate one of the methods proposed by mathematicians!

Three-Valued Logic

Note that in our system, we only have two states: heap and non-heap. Therefore, it is natural to add an intermediate state (intermediate heap) to solve the problem. For instance, we say 100 grains of sand make up an intermediate heap.

Can you identify the problems with this method?

Based on this idea of introducing more values to the system, can you come up with a method that improves upon the "third-valued logic method," and please identify the defect that this method modifies.

Do you find this paradox to be seemingly-trivial? If you do, don't be ashamed of admitting it because it does seem insignificant in the beginning, but, in fact, it is everywhere in our world.

Take color spectrum as an example. For any two colors that sit right next to each other on the spectrum, it is basically impossible to tell the difference, but it is also reasonable to not call color red "yellow," because there is an obvious distinction between the two colors. The problem, similar to the Heap Paradox, is that at which location of the color spectrum does the color change from red to yellow.

Can you think of any other real-life examples related to the Heap Paradox?

II. Liar Paradox

Assume that in a dichotomous city, Liars always lie, and Knights always tell the truth. If a resident claims that "I am lying," is the person a Liar or a Knight, and can you justify your answer?

Related topic: **Gödel's first incompleteness theorem**

III. Barber Paradox(Russell's Paradox)

1. Consider the following scenario: In a city of five people (Russell the barber, Allison the teacher, Rachel the cook, Billy the police, Maria the doctor), Russell only shaves for people who do not shave themselves, then can you come up with a set of customers that satisfy the standards?

   Note: Russell is the only one who can shave.

2. If some months later, Russell attends a university to study logic and changes his business standards. Now, he decides to shave **all** those and only those who do not shave themselves, then does Russell shave himself?

3. Can you find the difference between problem 1 and problem 2 and generalize the finding you get from previous question?