Logic Gates!

Early Elementary Math Circle

Computers are made out of devices called ”logic gates”. In this handout we’ll learn what logic gates are and what they do.

1 And gates

1. If you combine two true statements using the word ”and”, is the new statement true or false?

   For example, the statements ”grass is green” and ”the sky is blue” are both true. Is the following statement true or false?

$$\underbrace{\text{Grass is green}}_{\text{Statement 1}} \quad \textbf{and} \quad \underbrace{\text{the sky is blue.}}_{\text{Statement 2}}$$

2. If you combine a false statement and a true statement using the word ”and”, is the new statement true or false?

   For example, the statements ”grass is blue” is false while the statement”the sky is blue” is true. Is the following statement true or false?

$$\underbrace{\text{Grass is blue}}_{\text{Statement 1}} \quad \textbf{and} \quad \underbrace{\text{the sky is blue.}}_{\text{Statement 2}}$$

3. If you combine a true statement and a false statement using the word ”and”, is the new statement true or false?

   For example, the statements ”grass is green” is true while the statement ”the sky is green” is false. Is the following statement true or false?

$$\underbrace{\text{Grass is green}}_{\text{Statement 1}} \quad \textbf{and} \quad \underbrace{\text{the sky is green.}}_{\text{Statement 2}}$$

4. If you combine two false statements using the word "and", is the new statement true or false?

   For example, the statements "grass is blue" and "the sky is green" are both false. Is the following statement true or false?

   $$\underbrace{\text{Grass is blue}}_{\text{Statement 1}} \quad \textbf{and} \quad \underbrace{\text{the sky is green.}}_{\text{Statement 2}}$$

5. Fill in the following table.

Statement 1	Statement 2	Statement 1 **and** Statement 2
False	False	
False	True	
True	False	
True	True	

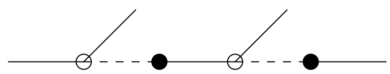


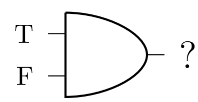
Figure 1: An “And gate” circuit. The current will go through only when *both* switches allow current to go through.

An ”And” gate is a device that takes two inputs and gives you one output. Each input has two possible values: ”True” and ”False”. The output has the value ”True” if the first input is True *and* the second input is True. Otherwise, the output has the value ”False”.

Figure **??** shows an ”And” gate’ made out of Legos. Notice the two inputs on the left. If an input rod is pushed inwards, then that input has a value of ”True”. If an input rod is not pushed inwards, then that input has a value of ”False”. In Figure **??**, both inputs have a value of ”False”. (In this picture, the number 0 is used as a short way of writing ”False”.)

Normally ”And” gates are made out of tiny devices called transistors. A real ”And gate” in a computer is so small that it can’t be seen without a powerful microscope. However, you can make them out of Legos also.

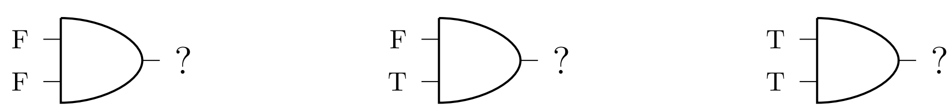
An "And" gate is drawn as follows:



In this picture, the first input has the value "True" ("T" for short) and the second input has the value "False" ("F" for short).

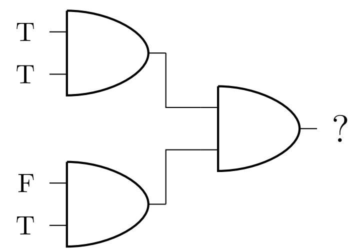
6. What is the output of the "And" gate in the picture above? Write the correct output on the picture.

7. What are the outputs of the following "And" gates?



8. Fill in the missing inputs on the following "And" gates:

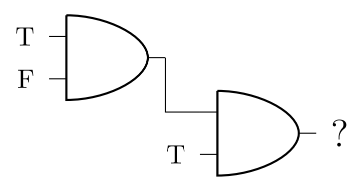
Multiple ”And” gates can be combined by feeding the output of one ”And” gate into the input of another ”And” gate. For example:



This combination of ”And ”gates is an example of a ”logic circuit”.

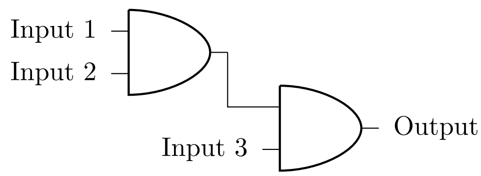
9. What is the output of the above logic circuit?

10. What is the output of the following logic circuit?



What values should the three inputs have in order for the output to have the value ”True”?

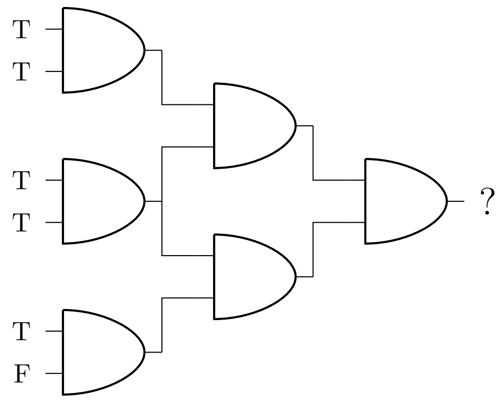
11. Fill in the table below with the correct output values for this logic circuit:



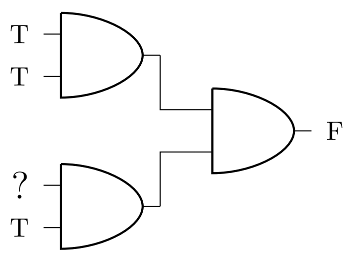
Input 1	Input 2	Input 3	Output
F	F	F	
F	F	T	
F	T	F	
F	T	T	
T	F	F	
T	F	T	
T	T	F	
T	T	T	

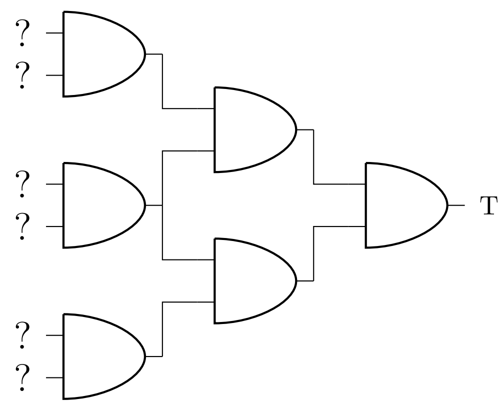
12. What is the output of the following logic circuit?

Start by finding the outputs of all of the "And" gates.



13. Fill in the missing inputs in the following logic circuits:





2 "Or" gates

1. If you combine a true statement and a false statement using the word "or", is the new statement true or false?

   For example, the statement "the sun is a star" is true, and the statement "the moon is a star" is false. Is the following statement true or false?

$$\underbrace{\text{The sun is a star}}_{\text{Statement 1}} \quad \textbf{or} \quad \underbrace{\text{the moon is a star.}}_{\text{Statement 2}}$$

2. If you combine a false statement and a true statement using the word "or", is the new statement true or false?

   For example, is the following statement true or false?

$$\underbrace{\text{The moon is a star}}_{\text{Statement 1}} \quad \textbf{or} \quad \underbrace{\text{the sun is a star.}}_{\text{Statement 2}}$$

3. If you combine a false statement and a false statement using the word "or", is the new statement true or false?

   For example:

$$\underbrace{\text{The moon is a star}}_{\text{Statement 1}} \quad \textbf{or} \quad \underbrace{\text{the sun is a planet.}}_{\text{Statement 2}}$$

4. If you combine two true statements using the word "or", is the new statement true or false?

   For example:

   The sun is a star (Statement 1) **or** the earth is a planet. (Statement 2)

Warning!

In other words, when mathematicians make a statement like "A is true or B is true", they mean that either A is true, or B is true, or *both* A and B are true.

This is not always the way people use the word "or" in everyday speech.

5. Fill in the following table.

Statement 1	Statement 2	Statement 1 **or** Statement 2
False	False	
False	True	
True	False	
True	True	

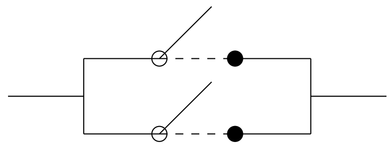
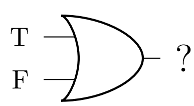


Figure 2: An "Or gate" circuit. The current will go through when *either* switch allows current to go through. In computers "Or" gates are made out of transistors and are extremely tiny, about 1/1000 the width of a human hair.

An "Or" gate is a device that takes two inputs and gives you one output. Each input has two possible values: "True" and "False". The output has the value "True" if either the first input is True *or* the second input is "True". (If both inputs are True, then the output is True – remember the warning above.) Otherwise, the output has the value "False".

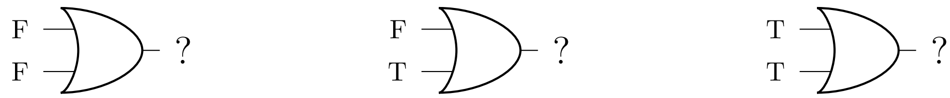
An "Or" gate is drawn as follows:



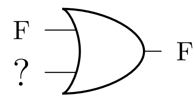
In this picture, the first input has the value "True" ("T" for short) and the second input has the value "False" ("F" for short).

6. What is the output of the "Or" gate in the picture above? Write the correct output on the picture.

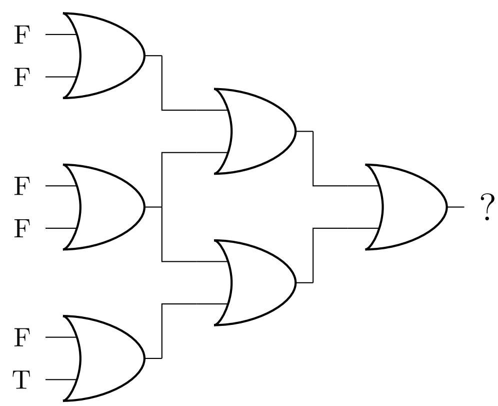
7. What are the outputs of the following "Or" gates?



8. Fill in the missing inputs on the following "Or" gates:



9. What is the output of the following logic circuit?



3 "Not" gates

1. If you modify a true statement using the word "not", is the new statement true or false? For example, the statement "ice is solid" is true. Is the following statement true or false?

   Ice is **not** solid.

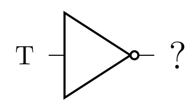
2. If you modify a false statement using the word "not", is the new statement true or false? For example, the statement "ice is liquid" is false. Is the following statement true or false?

   Ice is **not** liquid.

3. Fill in the blanks: When you modify a statement using the word "not", then true statements become _______ and false statements become _______ .

A "Not gate" is a device that takes *one* input and gives you one output. The input has two possible values: "True" and "False". The output has the value "True" if the input is "False", and the output has the value "False" if the input is "True".

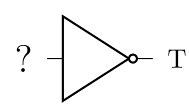
A "Not" gate" is drown as follows:



In this picture, the input has the value "True" ("T" for short)

4. What is the output of the "Not" gate in the picture above? Write the correct output on the picture.

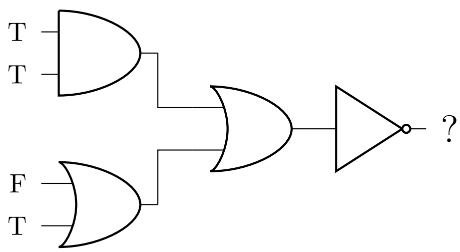
5. Fill in the missing input in the following "Not" gate:

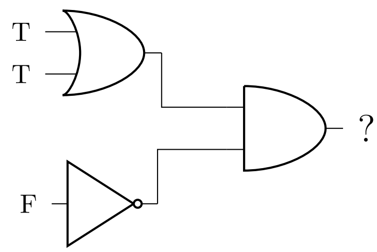


4 Logic Circuits

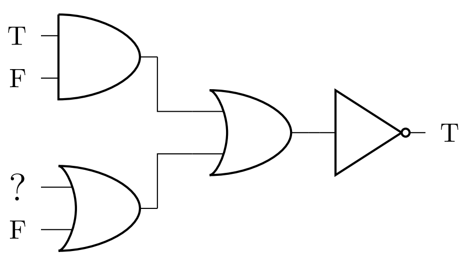
We can combine "And", "Or", and "Not" gates to make more complicated logic circuits.

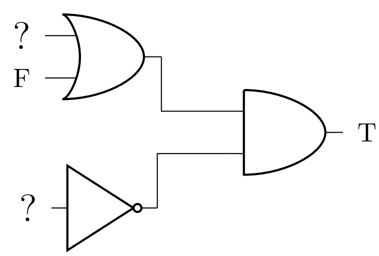
1. What are the outputs of the following logic circuits?

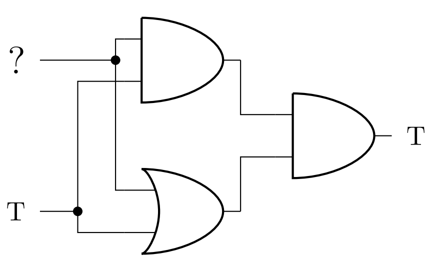




2. Fill in the missing inputs on the following logic circuits.







(In the last example, each input is fed into both an "And" gate and an "Or" gate.)

3. (*Challenge*) Design a logic circuit that has two inputs and one output, and obeys the following rules:

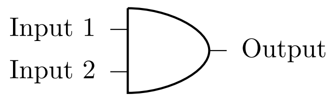
   - The output is "True" if exactly one (not both) of the inputs is true.
   - Otherwise, the output is "False".

   This logic circuit is called an "Exclusive Or" gate".

   Hint: a single input can be fed into more than one logic gate.

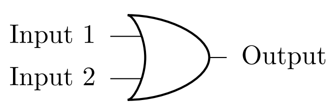
5 A "Universal" logic gate

1. Complete the table below, which completely describes the behavior of an "And" gate.



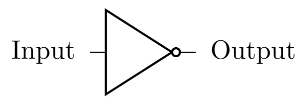
Input 1	Input 2	Output
F	F	?
F	T	?
T	F	?
T	T	?

2. Complete the table below, which completely describes the behavior of an "Or" gate.



Input 1	Input 2	Output
F	F	?
F	T	?
T	F	?
T	T	?

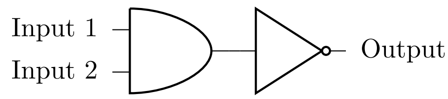
3. Complete the table below, which completely describes the behavior of a "Not" gate.



Input	Output
F	?
T	?

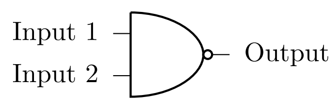
4. Design your own logic circuit in the space below. You can use "And", "Or", and "Not" gates. Make up inputs and figure out the output or outputs.

The logic circuit shown below is called a "Nand" gate:



"Nand" is short for "Not - And". A "Nand" gate consists of an "And" gate followed by a "Not" gate.

To save writing in the future, there is a special symbol for a Nand gate:

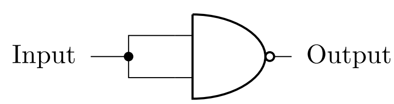


5. Complete the table below, which completely describes the behavior of a Nand gate.

Input 1	Input 2	Output
F	F	?
F	T	?
T	F	?
T	T	?

In the next few exercises, our goal is to show that any logic circuit can be made using only "nand" gates.

6. The logic circuit below has one input and one output:



(The Input value is fed into *both* inputs of the "Nand" gate.)

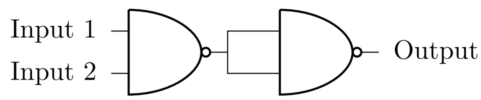
Fill in the table below, which describes the behavior of this logic circuit:

Input	Output
F	?
T	?

Compare this table with the table above that describes a "Not" gate. What do you notice?

Conclusion: A "Not" gate can be made out of a _____ gate !

7. Complete the table below, which describes the behavior of this logic circuit:



(The output of the first ”Nand” gate is fed into *both* inputs of the second ”Nand” gate.)

Input 1	Input 2	Output
F	F	?
F	T	?
T	F	?
T	T	?

Compare this table with the table describing an ”And” gate. What do you notice?

Conclusion: An ”And” gate can be made out of _____ gates!

8. (**Challenge**) How can you make an "Or" gate using only "Nand" gates?

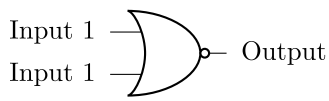
We have shown that "And", "Or", and "Not" gates can be made using only "Nand" gates.

This means that any logic circuit whatsoever can be made using only "Nand" gates!

For this reason, the "Nand" gate is said to be "Universal".

6 Nor gates

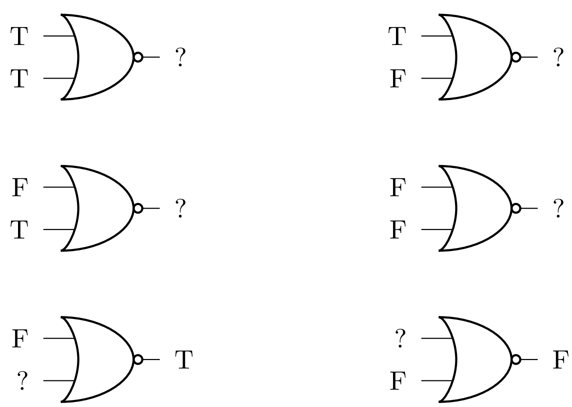
Another type of logic gate is called a "Nor" gate. A "Nor" gate has two inputs and one output, and is drawn on paper like this:



The output is "True" when neither Input 1 *nor* Input 2 is true. Otherwise, the output is "False".

In this section, we will show that the "Nor" gate is also universal!

1. Fill in the missing inputs and outputs in the following "Nor" gates.

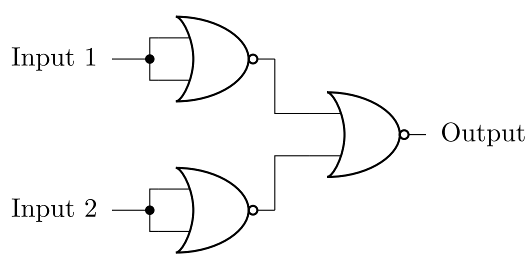


2. Draw a picture that shows how to make a "Nor" gate out of an "Or" gate and a "Not" gate:

3. Draw a picture that shows how to make a "Not" gate out of a "Nor" gate. (Hint: remember how a Not gate is made out of a "Nand" gate.)

4. Now that you know how to make a "Not" gate out of a "Nor" gate, draw a picture that shows how to make an "Or" gate out of "Nor" gates.

5. Complete the table below, which completely describes the behavior of the following logic circuit:



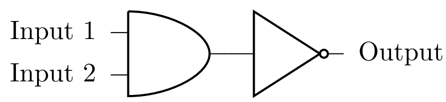
Input 1	Input 2	Output
F	F	?
F	T	?
T	F	?
T	T	?

Compare this table with the table that describes the And gate. What do you notice?

We have shown that a "Not" gate, an "Or" gate, and an "And" gate can be made out of _____ gates.

Conclusion: the "Nor" gate is universal!

6. Recall that a "Nand" gate is equivalent to an "And" gate followed by a "Not" gate:



   How can you make a "Nand" gate out of "Nor" gates?

7 Application: Outdoor security light

On the front porch of a house, there is a motion detection device that outputs "True" if it detects that someone is in the front yard, and "False" otherwise.

There is also a solar sensor which outputs "True" if it detects that the sun is shining, and "False" otherwise.

And finally there is a master switch, which outputs "True" if it's in the up position, and "False" otherwise.

An outdoor security light has a single input. The light turns on when the input has the value "True", and otherwise the light is off.

Design a logic circuit that connects the security light to the motion detector, the solar sensor, and the master switch, so that:

- The light turns on when someone is in the yard and the sun is not shining, or the master switch is in the up position.
- Otherwise, the light is off.

8 Adding numbers with logic gates

We usually write numbers using "decimal notation". In decimal notation, the number ten is written as 10.

However, it's also possible to write numbers using "binary notation". In binary notation, the number *two* is written as 10 .

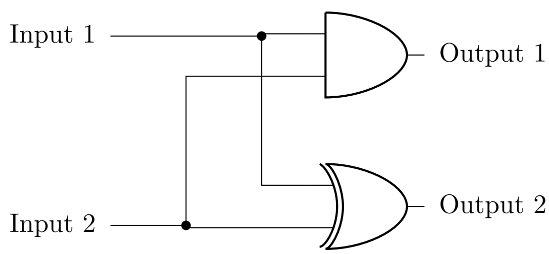
1. Complete the following addition problems. Write your answer using *binary* notation.

$$\begin{array}{r} 0 \\ +\ 0 \\ \hline \end{array} \qquad \begin{array}{r} 0 \\ +\ 1 \\ \hline \end{array} \qquad \begin{array}{r} 1 \\ +\ 0 \\ \hline \end{array} \qquad \begin{array}{r} 1 \\ +\ 1 \\ \hline \end{array}$$

We will learn more about binary notation at another time. For now, this is all we need to know.

2. Complete the table below, which describes the behavior of this logic circuit.

   But this time, we will use the number 0 as a short way of writing "False", and we will use 1 as a short way of writing "True".



Input 1	Input 2	Output 1	Output 2
0	0	?	?
0	1	?	?
1	0	?	?
1	1	?	?

   Compare this table with the addition problems you did on the previous page. What do you notice?

   This special logic circuit is the basic building block for more complicated logic circuits which are able to add large numbers.

9 Math Kangaroo questions

These questions are from the 2013 Canadian Math Kangaroo for 3rd and 4th graders.

1. Nancy bought 17 cones of ice-cream for her three children. Misha ate twice as many cones as Ana. Dan ate more ice-cream than Ana but less than Misha. How many cones of ice-cream did Dan eat?

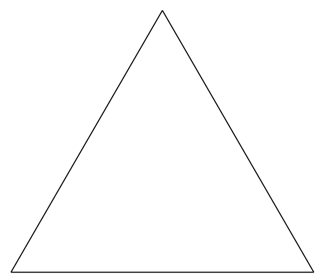
2. Cristi has to sell 10 glass bells that vary in price: 1 dollar, 2 dollars, 3 dollars, 4 dollars, 5 dollars, 6 dollars, 7 dollars, 8 dollars, 9 dollars, 10 dollars. Is it possible to divide all the glass bells into three packages so that all the packages have the same price?

3. In December Tom-the-cat slept for exactly 3 weeks. Which calculations should we do in order to find how many minutes he stayed awake during this month?

   (a) $(31 - 7) \times 3 \times 24 \times 60$
   (b) $(31 - 7 \times 3) \times 24 \times 60$
   (c) $(30 - 7 \times 3) \times 24 \times 60$
   (d) $(31 - 7) \times 24 \times 60$
   (e) $(31 - 7 \times 3) \times 24 \times 60 \times 60$

4. There are oranges, apricons and peaches in a big basket. How many fruits are there in the basket if the peaches and the apricots together are 18, the oranges and the apricots together are 28, and 30 fruits are not apricots?

5. In the shown triangle, first we join the midpoints of all the three sides. This way, we form a smaller triangle. We repeat this one more time with the smaller triangle, forming a new even smaller triangle, which we color in red. How many triangles of the size of the red triangle are needed to cover completely the original triangle, without overlapping?



6. Children in the school club had to arrange fitness balls according to their sizes from the biggest to the smallest one. Rebecca was comparing them and said: the red ball is smaller than the blue one, the yellow one is bigger than the green one, and the green one is bigger than the blue one. What is the correct order of the fitness balls?

7. In the Adventure Park, 20 children took part in two of the adventures. 15 of them participated in the "moving bridge" contest, and 20 of them went down the zip-wire. How many of the children took part in both adventures?

8. John is 33 years old. His three sons are $5, 6$ and 10 years old. In how many years will the three sons together be as old as their father?