Hyperbolic Models of Geometry

"Out of nothing I have created a strange new universe"

—János Bolyai

Euclid's Axioms:

1. Any two points can be joined with a line segment.
2. Any line segment can be extended indefinitely.
3. Given any straight line segment, a circle can be drawn having the segment as radius.
4. All right angles are congruent.
5. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines intersect each other on that side if extended far enough.

Instead of Axiom 5 we want our new geometry to have:

Axiom $5'$. Through any point not on a given line, there are at least two lines parallel to the first line that pass through the original point.

Question 0.1. (a) *Isometries* are symmetries that don't change lengths. What are the isometries of the Euclidean plane?

(b) Isometries form a group with composition as the multiplication. What are some subgroups of the Euclidean isometry group?

§1 Preliminaries: Inversive Geometry

Problem 1.1. If N is a point not on the circle ω and ℓ is a line through N intersecting ω at M and M', show that $|NM| \cdot |NM'|$ is independent of ℓ and equals $|NT|^2$, where $\vec{NT}$ is the ray tangent to ω.

Definition 1.2.

If ω has center O and radius r, then the inversion of a point M about ω is the point N on $\vec{OM}$ such that $|OM| \cdot |ON| = r^2$. We write this as $N = i_\omega(M)$.

The inversion maps O to the "point at infinity."

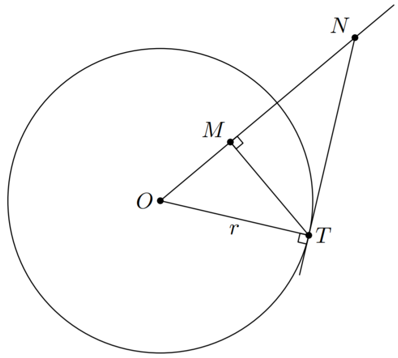

Figure 1: Inversion of M

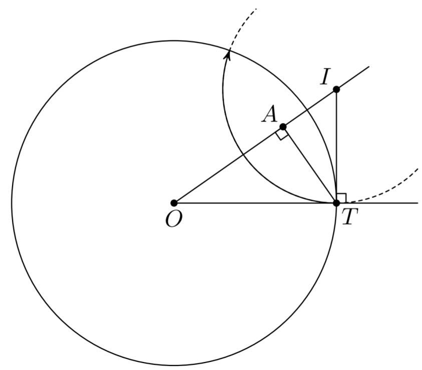

Figure 2: An inversion that takes A to O.

Problem 1.3. (a) Show that inversion about ω exchanges the interior and exterior of ω and fixes every point of ω.

(b) Show that circles orthogonal to ω are mapped to themselves under the inversion i_ω.

(c) What happens to circles (not necessarily orthogonal to ω) and lines under inversion? What happens if the circle or line passes through O?

Problem 1.4. Given a point A in the interior of ω, construct a circle orthogonal to ω so that A is mapped to O when inverted about this circle. See Figure 2.

Problem 1.5. Given two points A and B in the interior of ω, show that there is a unique circle orthogonal to ω (or a diameter) that passes through A and B. (Hint: use the previous problem)

Problem 1.6. Show that inversions preserves the measure of angles. Do the case of two intersecting lines.

Definition 1.7. For any four points A, B, C, and D the cross ratio is defied by

$$[A, B; C, D] = \frac{|AC| \cdot |BD|}{|AD| \cdot |BC|}$$

Problem 1.8. Show that the cross ratio $[A, B; C, D]$ is invariant under inversion, so long as the center of the circle of inversion is different than A, B, C, and D.

§2 The Poincaré Disc

The hyperbolic plane $\mathbb{H}^2$ can be modeled as the *Poincaré disc*, the unit ball in the complex plane

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

Points on the circle $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ are called *ideal points*. We define *hyperbolic lines* to be circles orthogonal the $\mathbb{S}^1$ or diameters. Lines are *parallel* if they do not intersect.

Definition 2.1. Let A, B be points in the Poincaré disc, and let P and Q be the ideal endpoints of the hyperbolic line joining A and B ordered such that $|AP| > |AQ|$ and $|BQ| > |BP|$. The *hyperbolic distance* between A and B is defined as

$$d(A, B) = \log\left([A, B; P, Q]\right)$$

Question 2.2. What are the isometries of the Poincaré disc (with respect to the hyperbolic distance)?

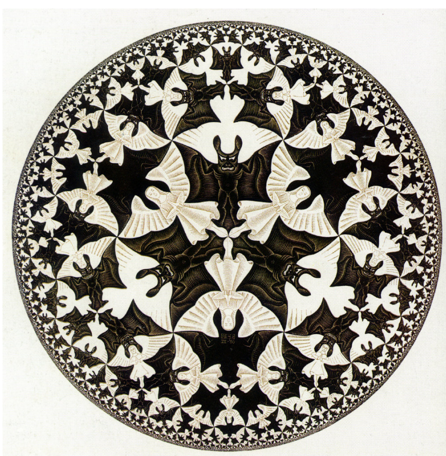

Figure 3: "Angel-devil" by M.C. Escher. Each devil has the same hyperbolic size!

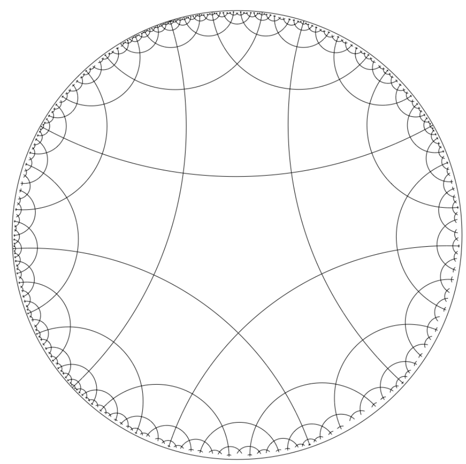

Figure 4: Tiling of hyperbolic plane by pentagons

Problem 2.3. (a) Show that $d(A, B) \geq 0$. When is the distance 0?

(b) Show that $d(A, B) = d(B, A)$

(c) If A, B, and C lie on the same hyperbolic line, show that $d(A, C) = d(A, B) + d(B, C)$. What happens if C is not on the same line?

(d) What is the hyperbolic distance between the origin and $\frac{1}{2}\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$?

(e) Find a general formula for $d(O, A)$ where A is Euclidean distance r from the origin. What happens when r is close to 0? When r is close to 1?

Problem 2.4. Show that Axiom $5'$ is satified with our new definition of "line."

Problem 2.5. (a) Sketch some hyperbolic triangles.

(b) Show that the sum of the angles of a hyperbolic triangle is less than π radians.

(c) What happens in the limiting case when the vertices of the triangle are ideal points?

Problem 2.6. Show that a Euclidean circle contained inside the Poincaré disc is also a hyperbolic circle. Where is the center of the hyperbolic circle?

Problem 2.7. (a) Which regular polygons can be used to tessellate the Euclidean plane (without gaps)?

(b) Figure 4 shows a tessellation of the hyperbolic plane by regular pentagons. Which regular polygons can be used to tessellate $\mathbb{H}^2$?

Definition 2.8. Given a line ℓ and a point P not on ℓ, there are rays r_1 and r_2 through P that are parallel to ℓ and such that any ray in the interior of $\angle(r_1, r_2)$ intersects ℓ. Drop a perpendicular from P to ℓ. The *angle of parallelism* is the angle between the perpendicular and r_1 (or equally, r_2).

Problem 2.9 (Lobachevskii's Theorem)**.** The angle of parallelism θ associated to P is related to the hyperbolic length d of the perpendicular by

$$e^{-d} = \tan\frac{\theta}{2}$$

Hint: Transform ℓ so that it is a diameter and $\ell \perp OP$. Draw the tangent to r_1 at P and look at the triangle formed with the endpoint of ℓ.

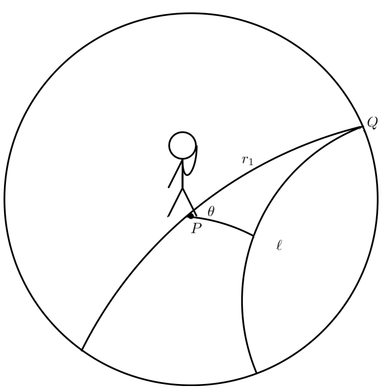

Figure 5: If a person living in the Poincaré disc was standing at a point P, that person can find out the distance to the line ℓ just by looking down the ray r_1 parallel to ℓ, calculating the angle θ, then using Lobachevskii's theorem. Strange!