Some polynomial equations (for example, \( x^2 + 1 = 0 \)) do not have solutions in the real numbers. Complex numbers were introduced in order to solve such equations. If we denote \( i = \sqrt{-1} \), then any complex number is of the form \( z = a + bi \), where \( a \) and \( b \) are real numbers. Complex numbers can be added and multiplied:

\[
(a + bi) + (c + di) = (a + b) + (c + d)i,
\]
\[
(a + bi)(c + di) = (ac - bd) + (ad + bc)i.
\]

From now on, we will see any complex number \( z = a + bi \) as a point in the plane whose coordinates are \( a \) and \( b \). This will enable us to solve geometry problems by using complex numbers. The absolute value \( |z| \) of \( z \) is by definition the distance between 0 and \( z \):

\[
|z| = \sqrt{a^2 + b^2}
\]

The argument \( \arg z \) of a non-zero complex number \( z \) is the counterclockwise angle between the line 0\( z \) (going through 0 and \( z \)) and the \( x \)-axis. Then \( z \) can be written as

\[
z = |z| (\cos(\arg z) + \sin(\arg z) i)
\]

The complex conjugate of \( z = a + bi \) is defined as \( \overline{z} = a - bi \). Using the complex conjugate we can divide by any non-zero complex number \( z \) by following the rule \( \frac{w}{z} = \frac{w\overline{z}}{|z|^2} \).

**Problem 1.** Let \( z, w \) be complex numbers.
(a) Prove that the distance between \( z \) and \( w \) is equal to \( |z - w| \).
(b) Prove that \( |z + w| \leq |z| + |w| \).
(c) Prove that 0, \( z \), \( z + w \), \( w \) form a parallelogram.

**Problem 2.** Show that if \( z, w \) are complex numbers, then

\[
|zw| = |z||w|, \quad \arg zw = \arg z + \arg w \text{ (modulo 360°)}.
\]

**Problem 3.** Let ABCD be a quadrilateral. Prove that

\[
AB - CD + AD - BC \geq AC - BC.
\]

**Problem 4.** Let \( z_1, z_2, z_3, z_4 \) be four distinct complex numbers. Prove that the lines \( z_1z_2 \) and \( z_3z_4 \) are perpendicular if and only if \( \frac{z_1 - z_2}{z_3 - z_4} \) is an imaginary number (that is, of the form \( bi \), for some real number \( b \)).

**Problem 5.** Construct two squares ABXY and ACZT in the exterior of a triangle ABC. Prove that the midpoint of the segment XZ is independent of A.

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Problem 6. Prove that four distinct points $z_1, z_2, z_3, z_4$ in the plane lie on a circle if and only if \( \frac{z_1 - z_3}{z_2 - z_3} / \frac{z_1 - z_4}{z_2 - z_4} \) is a real number.

Problem 7 (Ptolemy’s theorem). Show that if four points $A, B, C, D$ lie on a circle, in this order, then

\[
AB \cdot CD + AD \cdot BC = AC \cdot BC.
\]

Problem 8. Prove that three distinct points $z_1, z_2, z_3$ in the plane form an equilateral triangle if and only if $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$.

Problem 9. Let $ABC$ be a triangle. On the sides $AB, BC, CA$ consider points $X, Y, Z$ which divide the sides into the same ratio. Prove that the triangle $XYZ$ is equilateral if and only if the triangle $ABC$ is equilateral.

Problem 10 (Euler’s line). The circumcentre $O$, the centre of gravity (centroid) $G$ and the orthocentre $H$ of a triangle lie on the same line, in this order. Moreover, we have that $OH = 3OG$.

Problem 11 (Simson’s line). If $A, B, C$ are points on a circle, then the feet of perpendiculars from an arbitrary point $D$ on that circle to the sides of $ABC$ lie on a line.

Problem 12. Let $M$ and $N$ be interior points of the triangle $ABC$ such that $\widehat{MAB} = \widehat{NAC}$ and $\widehat{MBA} = \widehat{NBC}$. Prove that

\[
\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} = 1.
\]

Problem 13. On each side of a triangle construct equilateral triangles, lying exterior to the original triangle. Show that the centroids of the three equilateral triangles form themselves an equilateral triangle.

Problem 14. Let $A_1 A_2 \ldots A_n$ be a regular $n$-gon inscribed in a unit circle. Prove that the product of the distances from $A_1$ to all of the other $(n - 1)$ vertices is equal to $n$:

\[
A_1 A_2 \cdot A_1 A_3 \cdot \ldots \cdot A_1 A_n = n.
\]