Egyptian Fractions

One of the oldest surviving written works is the Rhind Papyrus, a scroll from ancient Egypt written almost 4000 years ago. It describes some of the mathematics known to the Egyptians, and one thing it talks about is how to write fractions. The Egyptians didn't express fractions the same way we do, using numerator and denominator. Instead, they expressed the fraction as a **sum of distinct unit fractions**.

For example, if an Egyptian wanted to talk about $3/4$, they would write it as $\frac{1}{2} + \frac{1}{4}$. Likewise, our Egyptian scribe would not write $6/7$, they would instead write $\frac{1}{2} + \frac{1}{3} + \frac{1}{42}$. It's important that the unit fractions are distinct; thus, the Egyptians would NOT express $2/5$ as $1/5 + 1/5$, but rather as a sum of *distinct* fractions of the form $1/k$. We call any representation of a number as a sum of distinct unit fractions an *Egyptian representation*, or *Egyptian fraction*, for that number.

1. How might the Egyptians write...
   (a) 5/6

   (b) 7/10

   (c) 7/12

   (d) 9/20

   (e) 4/5

   (f) 2/5

2. In our notation, 3/4 can also be written as 6/8, or 45/60, or indeed infinitely many different ways. Is there more than one way to write a number as an Egyptian fraction? Let's consider 3/4. We saw already that we can express $3/4$ as $1/2 + 1/4$.

   (a) Can you find a *different* pair of numbers k, ℓ such that $3/4 = 1/k + 1/\ell$? Find them, or explain why there is no other pair.

   (b) Try to find three distinct numbers m, n, p such that $3/4 = 1/m + 1/n + 1/p$.

   (c) Can you find three distinct numbers, r, s, t, *different* from those in part (b), such that $3/4 = 1/r + 1/s + 1/t$? Find them, or explain why there is no other triple.

3. Can you find *infinitely many* distinct triples u, v, w such that $3/4 = 1/u + 1/v + 1/w$? Describe such an infinite family, or explain why there can't be infinitely many triples.

4. (Bonus non-fractions puzzle!) Alyssa and King Tut run a 100-meter race, and Alyssa wins by 5 meters. To make it sporting, Alyssa lets King Tut start 5 meters in front of the starting line for the second race. Assuming they each run at the same speed as in the first race, who wins the second race, or is it a tie?

   In the third race, King Tut moves back to the regular starting line, but Alyssa starts 5 meters *behind* the starting line. Who wins this time, or is it a tie?

5. Does *every* fraction a/b have an Egyptian representation? And, can we always easily find one when it exists?

   Let's start with the second question. Think about how you found Egyptian representations in the previous problems (especially Problem 1). Is there some approach you took to organize your search, that might work in general to find Egyptian fractions?

   Try to think of some procedure which you can describe in a few words. It should consist of simple, familiar steps such as comparing fractions, adding and subtracting them, etc., and it should let you start with any fraction $a/b < 1$, perform some steps, and eventually lead to an Egyptian fraction representation of a/b.

   If you have an idea, try it out on a couple fractions to see if it works. Try to explain it to another student, or an assistant, and see if they understand and can do it themselves.

   Try for a few minutes to figure it out yourself or working with neighbors. If you need a hint, ask an assistant.

6. (a) Find a way to write the number 1 as a sum of three distinct unit fractions.

   (b) Explain why it's possible to write *any* unit fraction $1/k$ as a sum of three distinct unit fractions. (Hint: Use part (a).)

   (c) Given that every fraction has at least one Egyptian representation, explain why every fraction actually has *infinitely many* Egyptian representations.