

LAMC Junior Circle Solutions: Modular Arithmetic and Ciphers

Preston Carroll

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1. Warmup:

The Caesar cipher is a cipher in which every letter is shifted by a constant amount. For example, if I represent the word "CAB" with numbers it would be "2 0 1". Then, I would pick a number to shift all these numbers by; in our case let's shift it by 3. This means my new message or ciphertext will be "5 3 4". This is the same as "FDE". In order to decode this my partner would need to reverse this shift, that is, they would need to shift each of the letters by -3 to recover the word "CAB".

Let's try this with an example. The ciphertext of a message is provided below. The shift is +1:

Ciphertext: qmfbtf ipme vq uif ovncfs uisff xifo zpv bsf epof.

Plaintext: Please hold up the number three when you are done.

2. A powerful mathematical tool when working with ciphers is modular arithmetic. Modular arithmetic is similar to how we tell the time. For example, if it is 3PM right now, what time was it 4 hours ago? We do not say that it was -1PM, rather we say that it was 11AM. The way to deal with this mathematically is to write:

$$3 - 4 \text{ modulo } 12 = 11$$

The way to perform this calculation is very similar to division, except we are only interested in the remainder. So in the case of $31 \text{ mod } 26$ we would find that $\frac{31}{26} = 1$ remainder 5, and so

$$31 \text{ mod } 26 = 5$$

Let's try this with some examples. Patterns are very important in these type of problems and can make these calculations *much* easier to perform, so try to find some!

(a) $5 \text{ mod } 26 = 5$

(b) $71 \bmod 70 = \mathbf{1}$

(c) $101 \bmod 101 = \mathbf{0}$

(d) $\text{Googolplex} \bmod 2 = \mathbf{0}$

(e) $52 \bmod 20 = \mathbf{2}$

(f) $2018 \bmod 1000 = \mathbf{18}$

(g) $408 \bmod 7 = \mathbf{2}$

(h) $36972 \bmod 3 = \mathbf{0}$

3. We can apply this concept of modular arithmetic to more easily encrypt and decrypt our messages. Using your alphabet table, encrypt the following three messages with 3 different shifts of your choosing. One of them has been started for you.

(a) Ciphers are fun

Your shift:

Plaintext	c	i	p	h	e	r	s		a	r	e		f	u	n
Numbers	2	8	15	7	4	17	18		0	17	4		5	20	13
Shifted Numbers	3	9	16	8	5	18	19		1	18	5		6	21	14
Ciphertext	d	j	q	i	f	s	t		b	s	f		g	v	o

(b) Squircles are the best

Your shift:

Plaintext	s	q	u	i	r	c	l	e	s		a	r	e		t	h	e		b	e	s	t	
Numbers	18	16	20	8	17	2	11	4	18		0	17	4		19	7	4		1	4	18	19	
Shifted Numbers																							
Ciphertext																							

(c) Dodecahedron

Your shift:

Plaintext	d	o	d	e	c	a	h	e	d	r	o	n
Numbers	3	14	3	4	2	0	7	4	3	17	14	13
Shifted Numbers												
Ciphertext												

Answers May Vary

4. Inverses of Numbers

- (a) What is an inverse? If we have a number a , for example, and we want to find a^{-1} , we are really looking for the number such that when I multiply it by a , we get 1.

$$a \cdot a^{-1} = 1$$

So say I have 10. What is the inverse of 10? Well let's look at our definition, what number, when I multiply it by 10, do I get 1? That number is exactly $\frac{1}{10}$! Inverses in regular arithmetic are quite easy to find. Commonly we write the inverse of a number in the form of a^{-1} . So for if I am talking about the inverse of 10, I can write it as 10^{-1} , but to evaluate it I would write it as $\frac{1}{10}$

i. $2^{-1} = \frac{1}{2}$

ii. $13^{-1} = \frac{1}{13}$

iii. $\pi^{-1} = \frac{1}{\pi}$

iv. $1^{-1} = 1$

- (b) How is an inverse different with modular arithmetic? In actuality, inverses are the same in that they follow from the same definition, but finding them is generally more difficult to do by hand. Let's try and find the inverse of 9 modulo 26. We'd write this as

$$9^{-1} \text{ mod } 26$$

Looking back the definition of an inverse, we want to find a number such that when I multiply it by 9, I get 1. Keep in mind that we do have this extra step of doing our modular arithmetic and that we are only able to use whole numbers between 1 and 25. What is $1 \cdot 9 \text{ mod } 26$, well it is just 9, so 1 is not the inverse of 9 in mod 26. How about 2? $2 \cdot 9 \text{ mod } 26 = 18 \text{ mod } 26$. No luck again, let's try one more time. $3 \cdot 9 = 27 \text{ mod } 26 = 1 \text{ mod } 26$. Voila! 3 is the inverse of 9 mod 26, because when I multiply 3 by 9 in mod 26 I get 1.

- (c) Try and find the inverse of 7 in mod 17.

$$7^{-1} \text{ mod } 17 = 5$$

With some trial and error:

$$1 \cdot 7 \text{ mod } 17 \equiv 7$$

$$2 \cdot 7 \text{ mod } 17 \equiv 14$$

$$3 \cdot 7 \text{ mod } 17 \equiv 21 \text{ mod } 17 \equiv 4 \text{ mod } 17$$

$$4 \cdot 7 \text{ mod } 17 \equiv 28 \text{ mod } 17 \equiv 11 \text{ mod } 17$$

$$5 \cdot 7 \text{ mod } 17 \equiv 35 \text{ mod } 17 \equiv 1 \text{ mod } 17$$

So we can say that 5 is the modular inverse of 7 mod 17.

5. Introduction to Affine Ciphers:

Affine ciphers are very similar to the Caesar cipher with an extra twist. Rather than just adding one specific number to each letter in our message, we first multiply it by some number (mod 26) and then we add a number.

(a) Let's explore this with an example:

- Let's say we had the message "Hello World" which is "07-04-11-11-14 22-14-17-11-03" in numbers.
- Then we would pick a secret α and β . Our α is the number we will multiply by and β is the number we will add. I will pick $\alpha = 21$ and $\beta = 3$.
- To encode my message I will first do the modular multiplication so my first letter H, which is a 7, becomes $21 \cdot 7 = 147 \text{ mod } 26 = 17 \text{ mod } 26$.
- Then I will add my β of 3, so I add 3 to 17 to get 20 as the first letter of my cipher text (the letter U).
- If I continue this procedure for each letter I will get the ciphertext of: "UJAAL XLWAO"

(b) So how does this relate to modular inverses? Well, encryption seems pretty simple to perform if we know what α and β we want. But what about *decryption*? To perform the decryption we will need to reverse our modular multiplication by multiplying by the modular inverse. Let's pretend for a moment that we didn't know what the original plaintext was, and that we only had the ciphertext, α , and β . How do we go backwards? Well let's first take the letter U, convert it into its number form of 20, then subtract three. Now we have the number 17. All we know is that someone had a number, and then multiplied it by 21 in modulo 26 and got 17. To reverse this we can just use the modular inverse of 21! With some trial and error we can pretty quickly find that 5 is the inverse of 21 in mod 26. So we can just multiply 17 by 5 in mod 26 and get our original letter. $5 \cdot 17 \text{ mod } 26 = 85 \text{ mod } 26 = 7 \text{ mod } 26$. 7 is the number for H and we are done! To review:

$$\text{Ciphertext} = \alpha(\text{Plaintext}) + \beta \pmod{26}$$

$$\text{Ciphertext} - \beta = \alpha(\text{Plaintext}) \pmod{26}$$

$$\alpha^{-1}(\text{Ciphertext} - \beta) = \text{Plaintext} \pmod{26}$$

- Subtract β from the ciphertext
- Find the modular inverse of α and then multiply that by the number you obtained by subtracting β
- Convert back into letters.

$$\text{Ciphertext} = \alpha(\text{Plaintext}) + \beta \pmod{26}$$

$$20 = 21 \cdot (\text{Plaintext}) + 3 \pmod{26}$$

$$20 - 3 = 21 \cdot (\text{Plaintext}) \pmod{26}$$

$$5(17) = (\text{Plaintext}) \pmod{26}$$

$$85 = (\text{Plaintext}) \pmod{26}$$

$$7 \pmod{26} = H$$

(c) Homework: Perform a decryption of the following ciphertext:

(Hint $\alpha=9$, $\beta=3$)

Ciphertext: xs xj hbvo odaena sz envalis soxj xjq's xs?

Plaintext: It is much harder to decrypt this isn't it?

Try exploring more with this website: <https://cryptii.com/>

6. Which cipher is easier to work with?

Answers may vary, but generally the Affine cipher is more computationally intensive.

7. Which cipher is better to hide secrets from someone who can see your ciphertext, but does not know your key?

Both are very insecure since Eve could try all possibilities very quickly with a computer, but if the attacker is trying to crack these codes by hand, the affine cipher is stronger.