Lesson 7: Stable Matching

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Problem 1
Suppose there is a class of 3 girls: Alice, Beatrice, Cindy and 3 boys: Andy, Bob, Caleb.
Alice's preference is Bob > Andy > Caleb;
Beatrice's preference is Andy > Bob > Caleb;
Cindy's preference is Bob > Caleb > Andy;

Andy's preference is Alice > Beatrice > Cindy;
Bob's preference is Alice > Beatrice > Cindy;
Caleb's preference is Beatrice > Cindy > Alice.

Perform the Gale-Shapley algorithm on this setup and find out a stable matching of boys to girls.

Problem 2
Show that the Gale-Shapley algorithm on n boys and n girls terminates after at most n^2 steps.

Problem 3
Recall that we call a matching unstable if there are two pairs in the matching (m_1, w_1) and (m_2, w_2) such that m_1 prefers w_2 to w_1 and w_2 prefers m_1 to m_2. We call a matching stable if it is not unstable. Show that the Gale-Shapley algorithm returns a stable matching.

Problem 4
Consider a town with n men and n women seeking to get married to one another. Among $2n$ people there are k good women and k good men($1 \leq k \leq n-1$). Everyone would rather marry any good person than any bad person. i.e. the first k entries of everyone's preference list are the good people (of the opposite gender) in some order, and its next $n-k$ are the bad people (of the opposite gender) in some order. Show that in every stable matching, every good man is married to a good woman.

Problem 5
We say that a woman w is a valid partner of a man m if there exists a stable matching that contains the pair (m, w).

a) Show that any execution of Gale-Shapley algorithm results in the matching where every man is paired with his highest-ranked valid partner. We can conclude that every execution of the algorithm results in the same matching.

b) Show that any execution of Gale-Shapley algorithm results in the matching where every woman is paired with her lowest-ranked valid partner.