Infinity II

It's as easy as $\aleph_1, \aleph_2, \aleph_3, \ldots$

Math Circle

October 29, 2017

1. We aren't going to start off with the video this time, but I promise that we'll watch one soon. First, let's do some warm-up problems.

   (a) Can you find a bijection between the rational numbers (the fractions) and the natural numbers?

   (b) Can you find a graphical bijection between the points on the line segment $[0, 1]$ and $[0, 2]$? I know that you can find a formula to do it, but I want you to use a geometric argument.

(c) Using a picture, show that there is a bijection between the points in $[0, 1]$ and between any finite segment of the real line, that is $[a, b]$ for any $a, b \in \mathbb{R}, a < b$.

(d) Using a picture, can you find a bijection between all of the points on the real line and only the points in (0,1)?

(e) Do you think that there is a bijection between $\mathbb{R}$ and $\mathbb{R}^2 = (\mathbb{R}, \mathbb{R})$? If so try and come up with one. If not, explain (or better yet prove!) why not.

2. At this point you have probably seen time and time again that you can find bijections between lots and lots of different sets. The obvious question to ask here is, given two infinite sets, can you always find a bijection between then? Let's watch a video from Numberphile, called "Infinity is bigger than you think - Numberphile" with a surprising answer.

   `https://www.youtube.com/watch?v=elvOZm0d4H0`

   Let's digest that proof a little bit more. First of all, how can you prove that no such such bijection? Because you can't try all possible bijections, your only hope is to come up with an algorithm to 'automatically' rule out possible candidate bijections.

   The trick is this, suppose that we had a candidate for a bijection, then we could write it like this:

$$\begin{array}{ccl} \mathbb{N} & & [0,1) \\ 1 & \to & .032591\ldots \\ 2 & \to & .510539\ldots \\ 3 & \to & .000034\ldots \\ 4 & \to & .328641\ldots \\ 5 & \to & .889813\ldots \\ 6 & \to & .141599\ldots \\ \vdots & & \vdots \end{array}$$

   Now look at the $[0,1)$ column and focus on only the number along the diagonal,

$$\begin{array}{ccl} \mathbb{N} & & [0,1) \\ 1 & \to & .\underline{0}32591\ldots \\ 2 & \to & .5\underline{1}0539\ldots \\ 3 & \to & .00\underline{0}034\ldots \\ 4 & \to & .328\underline{6}41\ldots \\ 5 & \to & .8898\underline{1}3\ldots \\ 6 & \to & .14159\underline{9}\ldots \\ \vdots & & \vdots \end{array}$$

   Let m be the number in $[0,1)$ which is formed by taking every digit on the diagonal, so for this candidate bijection, $m = .010619\ldots$ Now, let $\tilde{m}$ be the number that you get if you add 1 to each digit of m modulo 10, so $\tilde{m} = .121720\ldots$ What can you say about $\tilde{m}$? Can it appear in the right hand side of the list? What does this tell you about this candidate bijection?

   (a) Use this to prove that there is no bijection between $\mathbb{N}$ and $[0,1)$. Just because this particular bijection maybe doesn't work, who cares?

(b) Prove that there is no bijection between $\mathbb{N}$ and $\mathbb{R}$. Because there is a function from $\mathbb{N} \to \mathbb{R}$, but not the other way around, we say that $|\mathbb{N}| < |\mathbb{R}|$.

(c) Prove if you have an infinite set $\mathcal{X}$, then $|\mathcal{X}| < |P(\mathcal{X})|$, where $P(\mathcal{X})$ is the power set of $\mathcal{X}$, in other words it is the set of all subsets of $\mathcal{X}$, infinite sets included. As an example, if $\mathcal{X} = \mathbb{R}$, then $P(\mathbb{R})$ would contain $\{1, 266, \pi, -e, 5\}$, $\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \dots\}$, and $\mathbb{R}$ itself.

(d) Do you think that there is a set $\mathcal{Y}$ such that $|\mathbb{N}| < |\mathcal{Y}| < |\mathbb{R}|$? Explain.

Challenge Questions

(a) Can you take an interval of length 1, cut it up into tiny pieces, and move those pieces around so that they covers every rational number? What if your initial interval is of length $\frac{1}{2}$? $\frac{1}{4}$? What does this tell you? Can you do the same thing for $\mathbb{R}$?

(b) Can you find a bijection between $\mathbb{R}$ and $\mathbb{R}^n$ where n can be any natural number?

(c) Can you find a bijection between $\mathcal{C}$ and $\mathbb{R}$, where $\mathcal{C}$ is the Cantor set? Ask an instructor to explain to you what the Cantor set is.

(d) Can you prove that $|\mathbb{R}| < |\mathcal{F}|$ where $\mathcal{F}$ is the set of all functions from $\mathbb{R}$ to $\mathbb{R}$?

(e) How many different sizes of infinities are there?

(f) Suppose that you have a countable number dials of that you can turn from (0,1). Each dial you can turn to as low as you want, but you **can't turn any of the dials to zero**. Prove that any given every number ϵ, you can adjust the dials so that the sum of the dials is less than ϵ.

(g) Can you do the same if you have an uncountable number of dials? Why or why not?