Voting Systems

High School Circle II

June 4, 2017

Today we are going to resume what we started last week. We are going to talk more about voting systems, are we are going to being our discussion by watching another video! This is the followup video to our discussion last week.

`https://www.youtube.com/watch?v=AhVR7gFMKNg`

Ok, that was a lot to take in. The goal for this worksheet is to make our way through the theorem, so that by the end everyone understands it fully. Before diving into the proof, we should be sure to define the assumptions properly. In order to do that, let's agree on some notation that we are going to use first.

Let $\mathbb{A} = \{A_1, A_2, \ldots, A_n\}$ be a list of candidates. Let's also assume that there are more than 2 candidates. Suppose that $\alpha = \{a_1, a_2, \ldots, a_m\}$ are a list the voters. The most natural way to define a person's vote would be to say something like: voter a_i's preferences $P(a_i)$ is given by: $P(a_i) = (A_{\sigma_1}, A_{\sigma_2}, \ldots, A_{\sigma_n})$ where $\sigma_1, \ldots, \sigma_n$ is a permutation of $1, \ldots, n$ (ask an instructor if you don't remember the definition of a permutation). If you knew $P(a_i)$ for every i, then you would know everyone's preferences.

Another way to talk about vote recording would be to do the following. For each person, ask them to tell you their favorite candidate out of every possible pair of candidates. For example, perhaps there are three candidates and you are questioning person a_i, then you would ask a_i if they preferred A_1 over A_2, then whom they preferred out of A_2 and A_3, and finally A_1 and A_3. Let's say that $P(a_i; A_j, A_k) = 1$ if person a_i prefers candidate A_j to A_k, and $P(a_i; A_j, A_k) = 0$ otherwise. Let's not worry about the case where $A_i = A_j$.

1. Your first task is to figure out of these two notations are really referring to the same thing. If they are, then if you know $P(a_i)$, then you should be able to figure out $P(a_i; A_j, A_k)$ for every j, k and vice versa.

   (a) First, some (very) easy examples, just to make sure that you were paying attention. Let $\mathbb{A} = \{A, B, C, D\}$ and $P(a) = (C, D, A, B)$. What is $P(a; D, A)$? $P(a; A, D)$? $P(a; A, C)$? $P(a; B, D)$?

(b) Now suppose that we recorded $P(b; A_j, A_k)$ in the following table. What is $P(b)$?

b	$P(b; A, \cdot)$	$P(b; B, \cdot)$	$P(b; C, \cdot)$	$P(b; D, \cdot)$
$P(b; \cdot, A)$		0	1	0
$P(b; \cdot, B)$	1		1	0
$P(b; \cdot, C)$	0	0		0
$P(b; \cdot, D)$	1	1	1	

(c) Is it true that you could convert every possible $P(a_i)$ into table of $P(a_i; A_j, A_k)$? Can you prove it?

(d) Is it true that you could convert every possible table $P(a_i; A_j, A_k)$ into a $P(a_i)$? Can you prove it? What if you assume that if $P(a_i; A_j, A_k) = 1$ then $P(a_i; A_k, A_j) = 0$ for every $j \neq k$?

(e) Suppose that we recorded $P(b; A_j, A_k)$ in the following table. What is $P(b)$?

b	$P(b; A, \cdot)$	$P(b; B, \cdot)$	$P(b; C, \cdot)$	$P(b; D, \cdot)$
$P(b; \cdot, A)$		0	0	1
$P(b; \cdot, B)$	1		0	0
$P(b; \cdot, C)$	1	1		0
$P(b; \cdot, D)$	0	1	1	

(f) We say that a relation is transitive if when $P(\cdot; A_i, A_j)$ and $P(\cdot; A_j, A_k)$ then we have also that $P(\cdot; A_i, A_k)$.

(g) Prove that if $P(\alpha_i; A_j, A_k)$ is transitive, then there is a corresponding $P(\alpha_i)$. What if it isn't transitive?

2. Now that we know that those two notations are equivalent, we can be confident that we can use whichever notation we prefer. To make the notation easier, let's represent $P(\alpha_i; A_j, A_k)$ as $A_j >_{\alpha_i} A_k$. And for the group's preference, let's use the notation $A_j > A_k$ without the subscript on the $>$.

   Now let's remember what Arrow's theorem says. It says that if a voting system respects unanimity, and independence of irrelevant alternatives, then it must be a dictatorship. Let's review those two definitions.

   - A voting system respects **unanimity** if whenever $A_j >_{\alpha_i} A_k$ for all i, then $A_j > A_k$ too.
   - A voting system has **independence of irrelevant alternatives** if the following is true. If $A_i > A_j$ for some collection of voter's preferences, then if a voter changes their preferences, but does not change their relative ranking of A_i and A_j, then $A_i > A_j$ for these new preferences too.

   (a) How does unanimity compare to the majority fairness criterion that we talked about last week? How are they related?

   (b) Out of the 5 voting schemes that we discussed last week, which ones respected unanimity? Prove it, or give a counterexample for each one.

(c) Can you think of two examples of voting schemes which don't respect unanimity?

(d) Out of the 5 voting schemes that we discussed last week, which ones had independence of irrelevant alternatives? Prove it, or give a counterexample for each one.

(e) Can you think of two examples of voting schemes which do have independence of irrelevant alternatives?

(f) Forget about the context of Arrow's theorem for a second. Do you think that unanimity and independence of irrelevant alternatives are good properties for a voting system to have? How would you convince someone who wasn't a mathematician that these are reasonable properties?

3. Now let's get some practice exploiting the property of independence of irrelevant alternatives.

   (a) Let's say that a groups preference for 5 candidates is: $A > B > C > D > E$ when voter a voted with preferences $C > A > B > E > D$. Then, suppose that voter a's preferences switch to $C > A > D > E > B$. What are all of the possible new group preferences if the voting scheme possesses independence of irrelevant alternatives?

(b) Prove the central lemma of arrow's theorem. That is, that if you have a polarizing candidate, then it will be either first, or last on the groups' preferences. *Hint, you need to use both unanimity and independence of irrelevant alternatives. If you don't use both, you didn't really prove it!

4. Ok, now let's move on to the meat of the theorem.

   (a) Let's move on to the 'test election' part of the video. Consider an election with n candidates $\mathbb{A} = \{A_1, A_2, \ldots, A_n\}$ and m voters. Now add a $n+1$ candidate Z. Prove that if Z is the last candidate on everyone ballot then it must be the last candidate in the group's preference, and likewise if it is first on every ballot.

(b) Start with Z at the bottom of everyone's ballot. Change a_1's ballot so that Z is at the top. Then change a_2's in the same way, then a_3's, etc... Prove that Z must at some point switch from being the last candidate to the first.

(c) Throughout this process, can Z flip back to being the bottom candidate? Why or why not?

(d) Let's say that b is a pivotal voter if b's vote is the first one to change Z from being at the bottom of the group preference to the top. We are going to now try and show that b ranks $A_1 >_b A_2$ (where b isn't A_1 or A_2) then the group preference must put put $A_1 > A_2$, no matter what the other voters do. Let's focus on the ballots just after b put Z at the top, and changes the group preference to Z. Let's pretend that b puts candidate A_1 above Z. Prove that $Z > A_2$.

(e) Prove that even if every other voter (besides b) changes their votes (while keeping Z in the same position) $Z > A_2$.

(f) Prove that in this case $A_1 > Z$.

(g) What property then proves that $A_1 > A_2$?

(h) Suppose now that candidate Z gets into a huge scandal, and everyone (including b) moves Z to the end of the vote. Is is still true that $A_1 > A_2$? Why or why not?

(i) Now suppose that Z drops out of the election entirely. Is is still true that $A_1 > A_2$?

(j) Extend this argument to show that if $A_i >_b A_j$, then $A_i > A_j$

(a) Bonus problem! No points for getting it correct, but you do get some pride. You are in the middle of a pool with a **very** angry goat standing on the boundary. This goat hates your guts. The pool is 10 feet in radius. You can swim at a rate of 1 foot per second, and the goat can run at a speed of 4 feet per second. You want to swim to the side of the pool and climb out, and the goat wants to headbutt you as soon as you do. If the goat is forced to run along the boundary, is it possible to swim in such a way that will let you get out of the pool without getting walloped? What about if the goat runs as 3 feet per second? 5? What is the fastest that the goat can run and still let you escape?