Problem 1. (i) Calculate the irrational numbers $\xi_1$ and $\xi_2$ represented by continued fractions corresponding to the sequences $2, 1, 2, 1, \ldots$ and $1, 3, 1, 2, 1, 2, \ldots$.

(ii) Find a general way of calculating the irrational number corresponding to a periodic continued fraction.
**Problem 2.** Let $\xi$ be a rational number. Show that there are only finitely many natural numbers $a, b$ with

$$0 < |\xi - \frac{a}{b}| < \frac{1}{b^2}.$$
Problem 3. For any real number $x$ show that

$$[m_0, m_1, \ldots, m_n, x] = [m_0, m_1, \ldots, m_n + \frac{1}{x}] .$$
Our first goal is to show that for every irrational number $\xi$ there are infinitely many solutions to the previous problem. First we need some set-up:

**Problem 4.** Let $\xi$ be an irrational number expressed by a continued fraction $[m_0, m_1, m_2, \ldots]$. Let $\frac{h_n}{k_n}$ be the rational number corresponding to the finite continued fraction $[m_0, m_1, \ldots, m_n]$.

(i) Calculate $h_i$ and $k_i$ for (at least) $i = 0, 1, 2, 3$.

(ii) Find formulas for $h$ and $k$ using $m_n$, $h_{n-1}$, $k_{n-1}$, $h_{n-2}$, and $k_{n-2}$ and prove them inductively. 
   *(Hint: the previous problem might help with the induction step.)*

(iii) Let $\xi_{n+1}$ be the irrational number corresponding to the continued fraction $[m_{n+1}, m_{n+2}, \ldots]$.
   Show
   $$\xi = \frac{\xi_{n+1} h_n + h_{n-1}}{\xi_{n+1} k_n + k_{n-1}}$$

(iv) Show that $k_{n+1} > k_n$.

(v) Show that we have $k_n h_{n+1} - k_{n+1} h_n = (-1)^n$. 


Problem 5.  (i) Show

\[ \frac{\xi - h_n}{k_n} = \frac{(-1)^n}{k_n(\xi_{n+1}k_n + k_{n-1})}. \]

(ii) With the notation from the previous problem we have

\[ |\xi - \frac{h_n}{k_n}| < \frac{1}{k_n k_{n+1}}. \]

(iii) Now let \( \xi \) be any irrational number. Show there are infinitely many natural numbers \( a, b \) with

\[ 0 < |\xi - \frac{a}{b}| < \frac{1}{b^2}. \]
Now we show that the continued fractions give the best possible rational approximation to irrational numbers:

**Problem 6.** We want to prove that for a irrational number $\xi$ given by a continued fraction $[m_0, m_1, m_2, \ldots]$ and for any pair of natural numbers $a, b$ the inequality

$$|\xi - \frac{a}{b}| < |\xi - \frac{h_n}{k_n}|$$

for some $n$ implies $b > k_n$.

(i) Explain how this statement means that continued fractions are the best rational approximation possible.

(ii) Show that the statement follows if we can prove that

$$|\xi b - a| < |\xi k_n - h_n|$$

implies $b \geq k_{n+1}$.
Problem 7. We will prove that

$$|\xi b - a| < |\xi k_n - h_n|$$

implies \(b \geq k_{n+1}\). We will proceed by contradiction and assume throughout the whole problem \(b < k_{n+1}\).

(i) Show that the system of linear equations \(x k_n + y k_{n+1} = b\) and \(x h_b + y h_{n+1} = a\) has integer solutions for \(x\) and \(y\).

(ii) Argue that \(x\) and \(y\) cannot be 0 and must have opposite sign.

(iii) Show

$$|\xi b - a| \geq |\xi k_n - h_n|$$

which is the final contradiction.