Problem 1. It takes a grandfather's clock 30 seconds to chime 6 o'clock. Assuming that the time of each chime is negligible compared to the time intervals between the chimes, how much time would it take the clock to chime 12?

\[ \begin{align*}
\text{Chimes} & \quad \text{Chime}\, 1 \quad \text{Chime}\, 2 \quad \text{Chime}\, 3 \quad \text{Chime}\, 4 \quad \text{Chime}\, 5 \quad \text{Chime}\, 6 \\
& \quad 5\, \text{seconds} \quad 5\, \text{seconds} \quad 5\, \text{seconds} \quad 5\, \text{seconds} \quad 5\, \text{seconds} \quad 5\, \text{seconds}
\end{align*} \]

* It takes 5 seconds per chime.

\[
\frac{5\, \text{seconds}}{\text{chime}} \cdot 12\, \text{chimes} = 60\, \text{seconds}.
\]

It would take 60 seconds to chime 12 o'clock.
Clock Arithmetic or a Circle as a Number Line

One way to turn a circle into a number line is to divide it into twelve equal parts. In this case, one step is usually called one hour.

0 coincides with 12. The hour hand moves from 0 to 1, from 1 to 2, ... from 11 to 12 just as it would have on the straight number line. However, 12 equals 0 on this circle, so there it goes again, from 1 to 2, and so on. We write down the fact that 12 equals 0 as

\[(0.1) \quad 12 \equiv 0 \pmod{12}\]

and read it as 12 is congruent to 0 modulo 12. The usual "=" sign is reserved for the straight number line; we use "≡" on the circle instead. The \(\pmod{12}\) symbol tells us that the circle is divided into 12 equal parts, so 12 coincides with 0, 13 – with 1, 14 – with 2, and so on. Or in the new notations,

\[13 \equiv 1 \pmod{12}, \ 14 \equiv 2 \pmod{12}, \ldots, 23 \equiv 11 \pmod{12},\]

\[24 \equiv 12 \equiv 0 \pmod{12}.\]
**Problem 2.** Divide the following numbers by 12, and write the remainders of the divisions.

\[ 15 \div 12 = 1 \text{ R } 3 \]  
\( \text{(because } 15 = (12 \times 1) + 3 \) 

\[ 21 \div 12 = 1 \text{ R } 9 \]

\[ 37 \div 12 = 3 \text{ R } 1 \]

\[ 46 \div 12 = 3 \text{ R } 10 \]

\[ 80 \div 12 = 6 \text{ R } 8 \]

Now, write down the modular congruencies for the following numbers.

\[ 15 \pmod{12} \equiv 3 \]

\[ 21 \pmod{12} \equiv 9 \]

\[ 37 \pmod{12} \equiv 1 \]

\[ 46 \pmod{12} \equiv 10 \]

\[ 80 \pmod{12} \equiv 8 \]

*Notice the relationships*

*These are remainders after division.*
**Problem 3.** Write down the modular congruencies for the following additions.

\[
9 + 4 \equiv 1 \pmod{12}
\]

\[
18 + 8 \equiv 2 \pmod{12}
\]

*Add the numbers (9 + 4 = 13), then take the mod (13 \mod 12 = 1 \mod 12)*

When subtracting two numbers ‘a’ and ‘b’, finding a - b means finding a number ‘c’ such that c + b = a.
For example, 5 - 3 = 2 because when you add 2 to 3, you get 5.

The same occurs with modular arithmetic.

Write down the modular congruencies for the following subtractions.

\[
8 - 3 \equiv 5 \pmod{12}
\]

\[
1 - 11 \equiv 2 \pmod{12}
\]

\[
4 - 15 \equiv 1 \pmod{12}
\]

*Subtract the numbers (8 - 3 = 5), then take the mod (5 \mod 12 = 5 \mod 12)*

*To find the mod of negative numbers (like -10 \mod 12), keep adding the mod base until you have a positive number.

Example:

\[
(-10) \mod 12 = (-10 + 12) \mod 12 = 2 \mod 12
\]

Add together → The number is positive, so stop
Another standard way to turn a circle into a number line is to divide it into 60 equal parts. Depending on the situation, the unit step is called either a minute or a second.

All the numbers living on this number line are considered modulo 60. In particular, $60 \equiv 0 \pmod{60}$. There are 60 minutes in an hour.

Problem 4.

$72 \equiv 12 \pmod{60}$

$135 \equiv 15 \pmod{60}$

$55 + 55 \equiv 0 \pmod{60}$

$-15 \equiv 45 \pmod{60}$

$240 - 59 \equiv 1 \pmod{60}$
Problem 5. An experiment in a biological lab starts at 7:00 AM and runs for 80 hours. What time will it end?

* Hint: Use 24 hour clock (where 1 pm = 13)

7 am = 7 on the 24 hour clock

7 + 80 = 87
87 mod 24 = 15

15 = 3 pm on 12-hour clock.

The experiment will end at 3 pm.

In the following problems, we will consider the mod 5 arithmetic, that of a circle divided into five equal parts.
Problem 6. Write down the modular congruencies for the following multiplication problems.

\[ 2 \times 3 \equiv 1 \pmod{5} \]  \hspace{1cm} \text{\textit{Multiply first}}

\[ 4 \times 4 \equiv 1 \pmod{5} \]  \hspace{1cm} \text{\textit{and then take}}

\[ 5 \times 7 \equiv 0 \pmod{5} \]  \hspace{1cm} \text{\textit{the mod of that number}}

\text{Ex: } 3 \times 3 = 6 \text{ then } \hspace{1cm} 6 \mod 5 = 1

Similar to how subtraction is related to addition, division is related to multiplication. Finding \( a \div b \) means finding a number 'c' such that \( b \times c = a \).

For example, \( 6 \div 3 = 2 \) because \( 3 \times 2 = 6 \).

Write down the modular congruencies for the following division problems.

\[ 1 \div 2 \equiv 3 \pmod{5} \]  \hspace{1cm} \text{\textit{Because } } 3 \times 2 = 6 \text{ and } 6 \mod 5 = 1

\[ 1 \div 3 \equiv 2 \pmod{5} \]

\[ 1 \div 4 \equiv 4 \pmod{5} \]  \hspace{1cm} \text{\textit{Because } } 4 \times 4 = 16 \text{ and } 16 \mod 5 = 1
Problem 7. Write down the modular congruencies using “mod (your age)”.

\[11 + 39 \equiv 4 \pmod{23}\]
\[22 + 17 \equiv 16 \pmod{23}\]
\[33 - 11 \equiv 22 \pmod{23}\]
\[2 - 50 \equiv 21 \pmod{23}\]
\[1 \times 10 \equiv 10 \pmod{23}\]
\[4 \times 4 \equiv 16 \pmod{23}\]

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