Practice problems

17th February 2006

Differentiate the following functions:

1. $y = 5x^2 + \frac{3}{\sqrt{x}} + \sqrt{x^5}$. $\quad y' = 10x - \frac{3}{2}\frac{1}{x\sqrt{x}} + \frac{5}{2}\sqrt{x^3}$
2. $y = x^{2/5} + 7x^{-8/3}$. $\quad y' = \frac{2}{5}x^{-3/5} - \frac{56}{3}x^{-11/3}$
3. $y = e^x \cdot x^5$. $\quad y' = e^x \cdot x^5 + 5e^x x^4$
4. $y = \ln x \cdot \cos x$. $\quad y' = \frac{\cos x}{x} - \ln x \cdot \sin x$
5. $y = e^{5\sqrt{x}}$. $\quad y' = \frac{5}{2}\frac{1}{\sqrt{x}} \cdot e^{5\sqrt{x}}$
6. $y = \cos(\tan(x))$. $\quad y' = -\sin(\tan x) \cdot \cancel{\csc^2(x)}$ $\sec^2(x)$
7. $y = \frac{\ln x}{\sqrt{x^2-5}}$. $\quad y' = \frac{\frac{\sqrt{x^2-5}}{x} - \ln x \cdot \frac{1}{\cancel{2}} \cdot \frac{1 \cdot \cancel{2}x}{\sqrt{x^2-5}}}{x^2-5} = \frac{(x^2-5) - x^2\ln x}{x(x^2-5)^{3/2}}$
8. $y = x^2 \cdot \cos(e^x)$. $\quad y' = 2x\cos e^x - x^2(\sin e^x) e^x$
9. $y = \sqrt{x^2-1} \cdot e^{\cos(\sqrt{x})}$. $\quad y' = \frac{1}{\cancel{2}\sqrt{x^2-1}} \cancel{2}x\, e^{\cos\sqrt{x}} + \sqrt{x^2-1} \cdot e^{\cos\sqrt{x}} \cdot (-\sin\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$
10. $y = \frac{x^2-3}{5^x}$. $\quad y' = \frac{2x \cdot 5^x - 5^x \ln 5 \cdot (x^2-3)}{5^{2x}}$

Find the derivative of an implicit function:

1. $x^2 + y^2 = 2xy$. $\quad 2x + 2y \cdot y' = 2y + 2x \cdot y' \Rightarrow y' = \frac{x-y}{x-y} = 1$
2. $y^2 x = x^3 + 3e^x$. $\quad 2y \cdot y'x + y^2\cancel{x} = 3x^2 + 3e^x \Rightarrow y' = \frac{3x^2 + 3e^x - y^2\cancel{x}}{2yx}$
3. $x^{2/3} + y^{2/3} = \ln x$. $\quad \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3} \cdot y' = \frac{1}{x} \Rightarrow y' = \frac{\frac{1}{x} - \frac{2}{3}x^{-1/3}}{\frac{2}{3}y^{-1/3}}$
4. $e^y x = e^x$. $\quad e^y \cdot y'x + e^y = e^x \Rightarrow y' = \frac{e^x - e^y}{x e^y}$

Find the derivative of the inverse function if $f(x)$ is given by

1. $f(x) = x^2 - 3$. $\quad \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{2x} = \frac{1}{2\sqrt{y+3}}$
2. $f(x) = 3e^{2x}$. $\quad \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{6e^{2x}} = \frac{1}{2y}$

Find the derivative $\frac{d}{dx}f^{-1}(a)$ for the following $f(x)$and a:

$f(2) = 2e^2$

1. $f(x) = xe^x$ and $a = 2e^2$. $\quad \frac{d}{dx} f^{-1}(2e^2) = \frac{1}{f'(2)} = \frac{1}{(e^x + xe^x)|_{x=2}} = \frac{1}{e^2+2e^2} = \frac{1}{3e^2}$
2. $f(x) = \sqrt{x} + x$ and $a = 2$. $\quad f(4) = 2 \quad \frac{d}{dx} f^{-1}(2) = \frac{1}{f'(4)} = \frac{-1}{(\frac{1}{2\sqrt{x}} - 1)|_{x=4}} = \frac{-1}{\frac{1}{4}-1} = +4/3$
3. $f(x) = x^2 \cos x$ and $a = (2\pi)^2$. $\quad \frac{d}{dx} f^{-1}((2\pi)^2) = \frac{1}{f'(2\pi)} = \frac{1}{(2x\cos x - x^2 \sin x)|_{x=2\pi}} = \frac{1}{4\pi}$

$f(2\pi) = (2\pi)^2$

Use logarithmic differentiation to find the derivatives of the following functions:

1. $x^{\ln x}$. $\quad \ln y = \ln x \cdot \ln x = (\ln x)^2, \ \frac{y'}{y} = 2\ln x \cdot \frac{1}{x} \Rightarrow y' = 2x^{\ln x} \cdot \frac{\ln x}{x} = 2x^{\ln x - 1} \ln x$
2. $(\cos x)^{\cos(x)}$. $\quad \ln y = \cos x \ln(\cos x), \ \frac{y'}{y} = -\sin x \cdot \ln(\cos x) + \frac{\cos x}{\cos x} \cdot (-\sin x) = -\sin x \cdot (1 + \ln(\cos x))$
3. $(\sin x)^{\sqrt{x}}$. $\quad \ln y = \sqrt{x} \cdot \ln(\sin x), \ \frac{y'}{y} = \sqrt{x} \cdot \frac{1}{\sin x} \cdot \cos x + \frac{1}{2\sqrt{x}} \ln(\sin x)$
4. $(\sqrt{x})^{e^x}$. $\quad \ln y = e^x \ln\sqrt{x}, \ \frac{y'}{y} = e^x \ln\sqrt{x} + \frac{1}{2\sqrt{x}} e^x \cdot \frac{1}{\sqrt{x}}$

Use linear approximations to find the approximate values of the following expressions:

$f(x) = \sqrt{x}, \ a = 49$

$f(x) = \ln x, \ a = e$

$f(x) = \cos x, \ a = \pi/2$

$f(x) = x^{25}, \ a = 1$

1. $\sqrt{48}$. $\quad \sqrt{48} \approx \sqrt{49} + \frac{1}{2\sqrt{49}} \cdot (-1) = 7 - \frac{1}{14} = 6\frac{13}{14}$
2. $\ln(1.1e)$. $\quad \ln(1.1e) \approx \ln e + \frac{1}{e} \cdot 0.1e = 1 + 0.1 = 1.1$
3. $\cos(\pi/2 + 0.5)$. $\quad \cos(\pi/2 + 0.5) \approx \cos\pi/2 + (-\sin\pi/2) \cdot 0.5 = -1/2$
4. $(0.99)^{25} \approx 1^{25} + 25x^{24} \cdot (-0.01) = 1 + 25 \cdot (-0.01) = 1 - \cancel{0.75} = \cancel{0.25}$

$0.25 \quad 0.75$

Approximate $f(x)$ by a linear approximation:

1. $f(x) = e^{2x+1}$ at $a = -1/2$. $\quad L(x) = 1 + 2 \cdot 1 \cdot (x + \frac{1}{2}) = 1 + 2(x + \frac{1}{2}) = 2x + 2$
2. $f(x) = \ln(1 + 2x)$ at $a = 0$. $\quad L(x) = 0 + \frac{1}{1+2\cdot 0} \cdot 2(x - 0) = 2x$

Some other problems:

1. Given that $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$, find the limit $\lim_{h \to 0} \frac{e^{ah} - 1}{bh}$, where a, b are some numbers.
2. Given that $f'(x) = 3x - 7\ln x$, compute the derivative $\frac{d}{dx} f(\cos(x))$. (*Hint:* use the chain rule).

1. $\lim_{h\to 0} \frac{e^{ah} - 1}{bh} = \lim_{h \to 0} \frac{e^{ah} - 1}{ah} \cdot \frac{a}{b} = 1 \cdot \frac{a}{b} = \frac{a}{b}.$

2. $\frac{d}{dx}(f(\cos x)) = \frac{df}{dx}(\cos x) \cdot (\cos x)' =$

$= -(3\cos x - 7\ln(\cos x)) \cdot \sin x$