MATH 3A
FIRST MIDTERM EXAMINATION

January 30th, 2006

Please show your work. Except on the multiple choice problems, you will receive little or no credit for a correct answer to a problem which is not accompanied by sufficient explanations. If you have a question about any particular problem, please raise your hand and one of the proctors will come and talk to you. At the completion of the exam, please hand the exam booklet to your TA. If you have any questions about the grading of the exam, please see the instructor *within 15 calendar days of the examination.*

Name:____________________________ Section:____________________

#1	#2	#3	#4	#5	Total

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Problem 1. Below, the correct answer are indicated:

1. Evaluate the limit of function $\lim_{x\to 3^-}[x-1]$, where $[a]$ denotes the integer value of a:
   (a) 1 – **CORRECT!**
   (b) 2;
   (c) 3;
   (d) does not exist.
   (e) none of the above.

2. Find the limit at infinity: $\lim_{x\to\infty} \frac{e^{-x}(x^2+2x-8)}{x^2-4}$:
   (a) 0 – **CORRECT!**
   (b) 1;
   (c) ∞;
   (d) $-\infty$;
   (e) none of the above.

3. Suppose that the size of a population at time t is given by the logistic curve $N(t) = \frac{600}{2+e^{-2t}}$. Find the carrying capacity K (i.e., the population size as $t \to \infty$) and the initial size $N(0)$ of the population:
   (a) $K = 400$ and $N(0) = 200$;
   (b) $K = 300$ and $N(0) = 100$;
   (c) $K = 300$ and $N(0) = 200$ – **Correct!**
   (d) $K = 200$ and $N(0) = 150$;
   (e) none of the above.

4. The Intermediate Value Theorem implies that the equation $\cos x = x$ has a solution on the interval
   (a) $(\pi, 2\pi)$;
   (b) $(0, \pi/2)$ – **Correct!**
   (c) $(-2\pi, -\pi)$;
   (d) on both $(\pi, 2\pi)$ and $(-2\pi, \pi)$;
   (e) none of the above.

Problem 2. Compute the limit

$$\lim_{x\to 0}\frac{x^2}{\sqrt{x^2+25}-5}$$

SOLUTION:

$$\begin{aligned}\lim_{x\to 0}\frac{x^2}{\sqrt{x^2+25}-5} &= \lim_{x\to 0}\frac{x^2}{\sqrt{x^2+25}-5}\cdot\frac{\sqrt{x^2+25}+5}{\sqrt{x^2+25}+5} = \\ \lim_{x\to 0}\frac{x^2(\sqrt{x^2+25}+5)}{(x^2+25)-25} &= \lim_{x\to 0}\frac{x^2}{x^2}(\sqrt{x^2+25}+5) = \lim_{x\to 0}(\sqrt{x^2+25}+5) = 10.\end{aligned}$$

Problem 3. Find all the values of $x \in \mathbb{R}$ for which the function

$$f(x) = \tan(\frac{x-1}{2})$$

is continuous.

SOLUTION: The function

$$f(x) = \tan(\frac{x-1}{2}) = \frac{\sin(\frac{x-1}{2})}{\cos(\frac{x-1}{2})}$$

is continous at all points where it is defined. It is defined iff the denominator is not equal to 0. We need to solve

$$\cos(\frac{x-1}{2}) = 0.$$

This means that $\frac{x-1}{2} = \pi k + \frac{\pi}{2}$ for any integer number k, i.e., for $k = 0, \pm 1, \pm 2, \ldots$. Therefore, $(x-1) = 2\pi k + \pi$, i.e., $x = 2\pi k + \pi + 1$.

Thus, the function is defined and continuous for all $x \in \mathbb{R}$ except for $x = 2\pi k + \pi + 1$ for all integer numbers k, i.e., the points $\ldots, -3\pi + 1, -\pi + 1, \pi + 1, 3\pi + 1, \ldots$.

Problem 4. Let

$$f(x) = \begin{cases} \frac{1}{x^2+1} & , \quad x \geq 1. \\ x^2 - ax & , \quad x < 1 \end{cases}$$

Find the value of a such that $f(x)$ is continuous for all values of $x \in \mathbb{R}$.

SOLUTION: The function $f(x)$ is continuous at all $x \neq 1$ independently of the value of a. At the point $x = 1$, $f(x)$ is continuous iff $\lim_{x\to 1^-} f(x) = f(1) = \lim_{x \to 1^+} f(x)$. This means that

$$\lim_{x\to 1}(x^2 - ax) = 1 - a = f(1) = \frac{1}{2},$$

i.e., $a = 1/2$. For all other values of a the function is discontinous.

Problem 5. Compute the following trigonometric limit:

$$\lim_{x\to 0} \frac{\sin(2x)}{\sqrt{x}\sin(3\sqrt{x})};$$

SOLUTION:

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(2x)}{\sqrt{x}\sin(3\sqrt{x})} &= \lim_{x\to 0} \frac{\sin(2x)}{2x} \cdot \frac{3\sqrt{x}}{\sin(3\sqrt{x})} \cdot \frac{2x}{3\sqrt{x}} \cdot \frac{1}{\sqrt{x}} = \\
&= \lim_{x\to 0} \frac{\sin(2x)}{2x} \cdot \lim_{x\to 0} \frac{3\sqrt{x}}{\sin(3\sqrt{x})} \cdot \frac{2}{3} = \frac{2}{3}.
\end{aligned}$$