Homework #3 solutions

O'Connor

3.3.1, p.142: 6, 8, 19, 28
3.4.1, p.148: 2, 4, 10, 11, 14, 17
3.5.3, p.152: 2, 4, ~~8~~, ~~9~~

3.3.1

6) Degree of denominator is greater than degree of numerator, So the limit is 0.

8) Degree of numerator = degree of denominator. So the limit is the ratio of the coefficients of the leading terms in the numerator and denominator. (See p. 140.) So the limit is $\frac{-1}{-2} = \frac{1}{2}$.

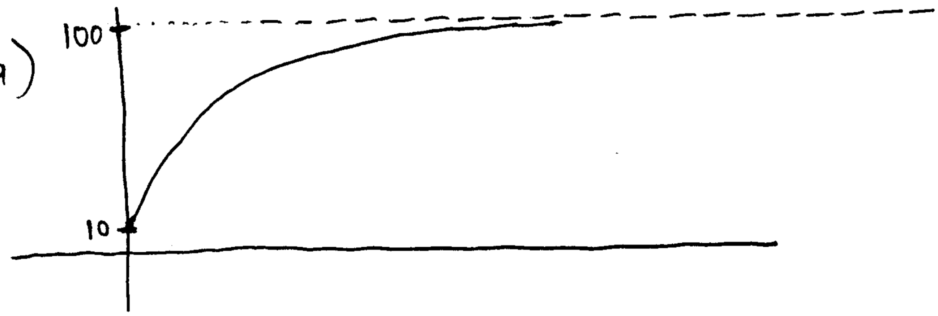
19) $-1 \le \sin(x) \le 1$.

e^{-x} is always positive, so we can multiply through by e^{-x} and get $-e^{-x} \le e^{-x}\sin(x) \le e^{-x}$.

e^{-x} and $-e^{-x}$ both approach 0 as $x \to \infty$.

By the sandwich theorem, $e^{-x}\sin(x) \to 0$ as $x \to \infty$.

28) a)



b) $$\lim_{t\to\infty} \frac{100}{1+9e^{-t}} = \frac{\lim_{t\to\infty} 100}{\lim_{t\to\infty}(1+9e^{-t})}$$

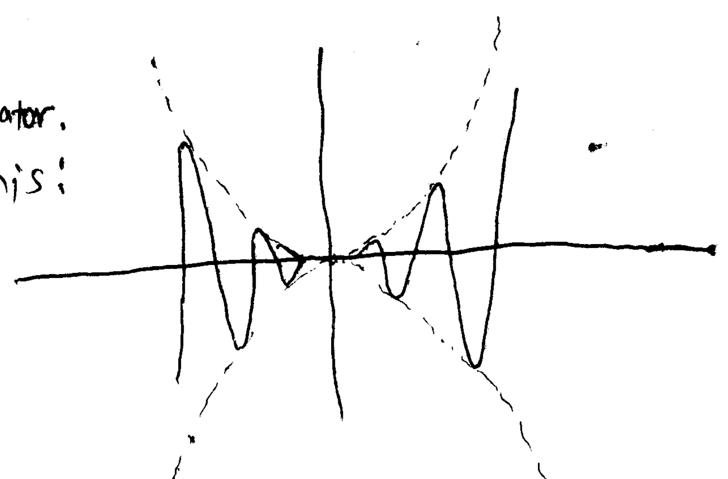
$$= \frac{100}{\lim_{t\to\infty} 1 + \lim_{t\to\infty} 9e^{-t}} = \frac{100}{1+9\left(\lim_{t\to\infty} e^{-t}\right)}$$

$$= \frac{100}{1+9\cdot 0} = \boxed{100}.$$

This agrees with the graph in part a).

Section 3.4.1

2) a) Just use calculator.
Something like this:



(#2 continued)

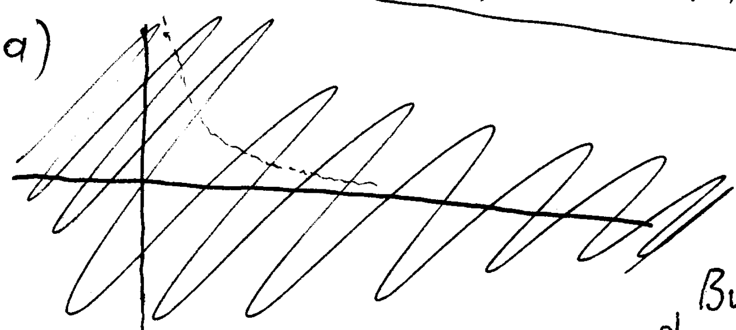
b) For any $x \neq 0$, $-1 \leq \sin(1/x) \leq 1$.

x^2 is nonnegative, so we can multiply through by x^2 to get

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2.$$

c) $-x^2 \to 0$ as $x \to 0$. $x^2 \to 0$ as $x \to 0$.
So by Sandwich theorem, $x^2 \sin(1/x) \to 0$.

4 a)



NOTE:
$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1.$$
But in this problem we show $\lim_{x \to \infty} \frac{\sin(x)}{x} = 0$.

Just use graphing calculator.

b) $\lim_{x \to \infty} \sin(x)$ does not exist. $\lim_{x \to \infty} x$ does not exist.

So, we can't say $\lim_{x \to \infty} \frac{\sin(x)}{x} = \dfrac{\lim_{x \to \infty} \sin(x)}{\lim_{x \to \infty} x}$.

The "basic rules for finding limits" don't apply in this situation.

c) If $x > 0$, $-1 \leq \sin(x) \leq 1$. Dividing through by x gives $\frac{-1}{x} \leq \frac{\sin(x)}{x} \leq 1/x$. $\frac{-1}{x} \to 0$ and $\frac{1}{x} \to 0$ as $x \to \infty$. So $\frac{\sin(x)}{x} \to 0$ as $x \to \infty$.

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(10) Let $z = \frac{-\pi x}{2}$. (As $x \to 0$, $z \to 0$ also.)

Then $2z = -\pi x$, so $\frac{2z}{-\pi} = x$, so $2x = 4z/-\pi$.

$$\lim_{x\to 0} \frac{\sin(-\pi x/2)}{2x} = \lim_{z\to 0} \frac{\sin(z)}{(4z/-\pi)} = \lim_{z\to 0} \frac{-\pi}{4} \frac{\sin(z)}{z}$$

$$= \frac{-\pi}{4} \lim_{z\to 0} \frac{\sin z}{z} = \frac{-\pi}{4} \cdot 1 = \boxed{-\pi/4.}$$

(11) I would like to somehow make this look more like the limit as $x \to 0$ of $\frac{\sin(x)}{x}$, which is a limit I can handle. The factor of π doesn't worry me much. But that square root on the bottom disturbs me. Can I get rid of it??

Aha! If I multiply top and bottom by $\sqrt{x}$, the bottom becomes just x. That seems promising.

$$\lim_{x\to 0} \frac{\sin(\pi x)}{\sqrt{x}} = \lim_{x\to 0} \frac{\sqrt{x}\sin(\pi x)}{\sqrt{x}\sqrt{x}} = \lim_{x\to 0} \sqrt{x} \frac{\sin(\pi x)}{x}$$

$$= \left(\lim_{x\to 0} \sqrt{x}\right) \cdot \left(\lim_{x\to 0} \frac{\sin(\pi x)}{x}\right) = 0 \cdot \lim_{x\to 0} \frac{\sin(\pi x)}{x}.$$

Let $z = \pi x$, (so $x = z/\pi$.) Then $\lim_{x\to 0} \frac{\sin(\pi x)}{x} = \lim_{z\to 0} \frac{\sin z}{(z/\pi)}$

$= \pi \cdot \lim_{z\to 0} \frac{\sin(z)}{z} = \pi \cdot 1 = \pi$. So, final answer is $0 \cdot \pi = \boxed{0}$.

(14) A way that doesn't work:

$1-\cos^2 x$ is a difference of two squares, so it is crying out to be factored. $\dfrac{1-\cos^2 x}{x^2} = \dfrac{(1-\cos x)(1+\cos x)}{x^2} = \left(\dfrac{1-\cos x}{x}\right)\left(\dfrac{1+\cos x}{x}\right)$.

This seems promising, because $\dfrac{1-\cos x}{x}$ as $x \to 0$ is one of the special trig limits we know. But alas, $\lim\limits_{x\to 0} \dfrac{1+\cos x}{x}$ does not exist, so we can't use the basic limit laws. Perhaps we should try something else.

A way that works:

Whenever we see $1-\cos^2(x)$, we should be tempted to replace it with $\sin^2(x)$. ($\sin^2 x + \cos^2 x = 1$, so $1-\cos^2 x = \sin^2 x$.) Doing so reduces this problem to example 3 b), p. 147.

$$\lim_{x\to 0} \frac{1-\cos^2 x}{x^2} = \lim_{x\to 0} \frac{\sin^2 x}{x^2} = \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2$$

$$= \left(\lim_{x\to 0} \frac{\sin x}{x}\right)^2 = 1^2 = \boxed{1.}$$

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(17) $\lim_{x \to 0} \frac{\sin x (1-\cos x)}{x^2} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{(1-\cos x)}{x}$

$= \left(\lim_{x \to 0} \frac{\sin x}{x}\right)\left(\lim_{x \to 0} \frac{1-\cos x}{x}\right) = 1 \cdot 0 = 0.$

Section 3.5.3

(2) a) Just use graphing Calculator.

b) $f(-3) = (-3)^3 - 2(-3) + 3 = -18 < 0.$

$f(-1) = (-1)^3 - 2(-1) + 3 = -1 + 2 + 3 = 4 > 0.$

f is continuous everywhere.

So IVT tells us $f(x) = 0$ for at least one value of x in $(-3, -1)$

(4) a) Just use graphing calculator.

b) Let $f(x) = \sin(x) - x.$

$f(1) = \sin(1) - 1 \approx .84 - 1 < 0.$

$f(-1) = \sin(-1) - (-1) = \sin(-1) + 1 = -\sin(1) + 1$ (because $\sin(-x) = -\sin(x)$ for any x)

$= 1 - \sin(1) \approx 1 - .84 > 0.$

f is continuous everywhere.

So, IVT tells us $f(x) = \sin(x) - x = 0$ for at least one value of x in $(-1, 1).$

If $\sin(x) - x = 0$, that means $\sin(x) = x.$

So $\sin(x) = x$ has a solution in $(-1, 1).$