MATH 3A
FINAL EXAMINATION

March 21st, 2006

Please show your work. Except on the multiple choice problems, you will receive little or no credit for a correct answer to a problem which is not accompanied by sufficient explanations. If you have a question about any particular problem, please raise your hand and one of the proctors will come and talk to you. At the completion of the exam, please hand the exam booklet to your TA.

Name:____________________ Section:____________________

#1	#2	#3	#4	#5	#6	Total
#7	#8	#9	#10	#11	#12	

Problem 1. Find all the values of $x \in \mathbb{R}$ for which the function $f(x) = e^{-\sqrt{\frac{x}{x+1}}}$ is continuous.

$f(x)$ is continuous iff $\frac{x}{x+1} \geqslant 0$

$$\frac{x}{x+1} \geqslant 0 \iff \begin{cases} x \geqslant 0, & \text{if } x+1>0 \\ x \leqslant 0 & \text{if } x+1<0 \end{cases} \iff$$

$$\iff \begin{cases} x \geqslant 0, \text{ if } x > -1 \\ x \leqslant 0, \text{ if } x < -1 \end{cases} \iff$$

$$\iff \begin{cases} x \geqslant 0 \\ x < -1 \end{cases}$$

Answer: $f(x)$ is continuous for $x < -1$ and $x \geqslant 0$.

Problem 1 is similar to # 2, p. 137 (hw 2) and Practice Midterm 1, problem 4(a) – (d).

Problem 2. Evaluate the limit $\lim_{x \to 0} x^2 \cdot \cos\left(e^{1/x}\right)$.

$$-1 \leq \cos\left(e^{1/x}\right) \leq 1$$

as $x \to 0$, $$-x^2 \leq x^2 \cdot \cos\left(e^{1/x}\right) \leq x^2$$

$$\underset{0}{\downarrow} \qquad\qquad \underset{0}{\downarrow}$$

By Sandwich Thm, $\lim_{x \to 0} x^2 \cdot \cos(1/x) = 0$.

(Note: $\lim_{x \to 0} \cos\left(e^{1/x}\right)$ does not exist.

The function $\cos\left(e^{1/x}\right)$ oscillates near $x = 0$.

So, Sandwich thm has to be used to evaluate this limit.)

Problem 2 is similar to # 10, 11, 14, 17, p. 148 (hw 3,

Problem 3. Use the formal definition of the derivative to find the derivative of $y = x^2$.

$$\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{2hx + h^2}{h} =$$

$$= \lim_{h\to 0} (2x+h) \quad = 2x.$$

Problem 3 is similar to #13, 14, 27, 29, p. 177 (hw 4)

Problem 4. Find all points on the graph of $y = 2x^3 - 4x + 1$ at which the tangent line to the graph is parallel to the line $y - 2x = 1$.

The slope of the tangent line is given by the derivative:

$y' = 6x^2 - 4$

The slope of $y = 2x + 1$ is 2.

Tangent line is parallel to this line iff the slopes are equal:

$$6x^2 - 4 = 2$$

$$6x^2 = 6$$

$$x^2 = 1$$

$$x = \pm 1.$$

Problem 4 is similar to # 46, 53, 66, p. 185 (hw 5)

Problem 5. Let $f(x)$ be a function that is differentiable for all x. Find an expression for the derivative of $y = (f(x))^2 + f(x^2)$ in terms of $f(x)$ and $f'(x)$.

$$y' = 2 \cdot f(x) \cdot f'(x) + f'(x^2) \cdot 2x$$

Problem 5 is similar to problem 2 at the bottom of page 2 of practice midterm 2; problem # 23, p. 243 (see practice list for final exam).

Problem 6. Find the derivative of the function $y = \ln(x \sin(xe^x))$

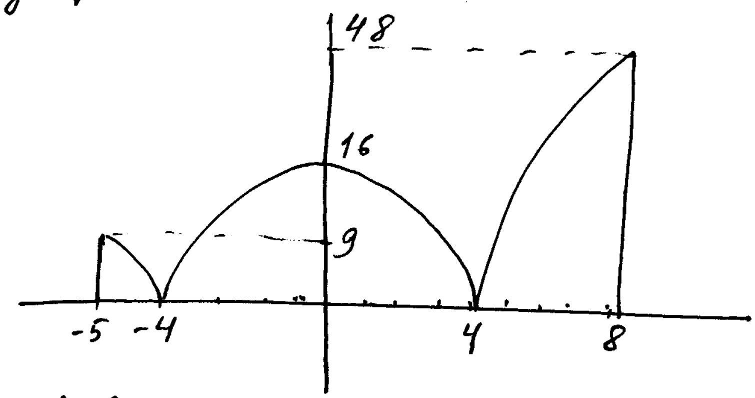
$$y' = \frac{1}{x \cdot \sin(xe^x)} \cdot \left(\sin(xe^x) + x \cdot \cos(xe^x) \cdot (e^x + x \cdot e^x) \right).$$

Similar to problems from section 4.4.5. (p. 208-209)

Similar to a problem done in class (first week of March
and # 4, p.259 (hw 8))

Problem 7. Find the (global) maximum and minimum values of the function $y = |16 - x^2|$ on the interval $-5 \le x \le 8$.

Method 1: Graph the function:



global max: $x = 8$, $y(8) = 48$.
global min: $x = \pm 4$, $y = 0$.

Method 2:

$$y = \begin{cases} 16 - x^2, & |x| \le 4 \\ x^2 - 16, & -5 \le x < -4 \text{ and } 4 < x \le 8 \end{cases}$$

$$y' = \begin{cases} -2x, & -4 \le x \le 4 \\ 2x, & -5 \le x < -4 \quad \text{and} \quad 4 < x \le 8. \end{cases}$$

1. $y' = 0$ for $x = 0$. y' changes sign from - to +. Thus, $x = 0$ is a pt of local maximum. $y(0) = \underline{16}$.
2. y' does not exist at $x = \pm 4$. $y(\pm 4) = \underline{0}$.
3. Endpoints: $y(-5) = 9$, $y(8) = 48$.

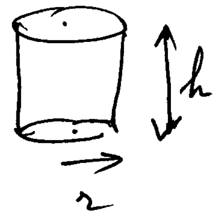
Comparing results in ①-③, we get: global max: $x = 8$, $y(8) = 48$
global min: $x = \pm 4$, $y = 0$.

Similar to # 15 (hw 10) p. 297) and identical to #16 (see final exam practice problem list)

Problem 8. Find the dimensions of a right-circular cylindrical can that is open on the top and closed on the bottom, so that the can holds one liter and uses the least amount of material.

Volume: $V = \pi r^2 \cdot h = 1$

Surface Area: $S = \pi r^2 + 2\pi r \cdot h$

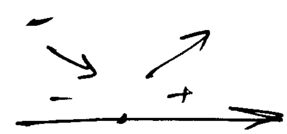


$h = \dfrac{1}{\pi r^2}$

$S(r) = \pi r^2 + 2\pi r \cdot \dfrac{1}{\pi r^2} =$

$= \pi r^2 + \dfrac{2}{r}.$

$S'(r) = 2\pi r - \dfrac{2}{r^2} = 0. \Rightarrow \dfrac{2(\pi r^3 - 1)}{r^2} = 0 \Rightarrow r = \dfrac{1}{\sqrt[3]{\pi}}$



$r_0 = \dfrac{1}{\sqrt[3]{\pi}}$

$\Rightarrow r_0 = \dfrac{1}{\sqrt[3]{\pi}}$ is

a pt of local min.

$h = \dfrac{1}{\pi r^2} = \dfrac{1}{\pi \sqrt[3]{\pi^2}} = \dfrac{1}{\pi^{5/3}}$

Similar to graphs in the practice final exam.
& # 35, 37 (p. 287-288) (hw 9).

Problem 9. Let $y = \dfrac{x^2 - 1}{x}$.

(a) Determine where this function is increasing and decreasing; find all points of local minimum and maximum, as well as the values of the function at those points.

$y' = 1 + \dfrac{1}{x^2} > 0$ for all x;

y is increasing everywhere.

(b) Determine where the function is concave up and down and find all inflection points.

$y'' = -\dfrac{2}{x^3}$

$\begin{cases} y'' > 0 & \text{for } x < 0 \quad \text{- concave up} \\ y'' < 0 & \text{for } x > 0 \quad \text{- concave down} \end{cases}$

(c) Determine all asymptotes.

Vertical: $\lim\limits_{x \to 0^+} \left(x - \dfrac{1}{x}\right) = -\infty$

$\lim\limits_{x \to 0^-} \left(x - \dfrac{1}{x}\right) = +\infty.$

Oblique: $\lim\limits_{x \to +\infty} (y - x) = \lim\limits_{x \to +\infty} \left(-\dfrac{1}{x}\right) = 0^-$

$\lim\limits_{x \to -\infty} (y - x) = \lim\limits_{x \to -\infty} \left(-\dfrac{1}{x}\right) = 0^+$

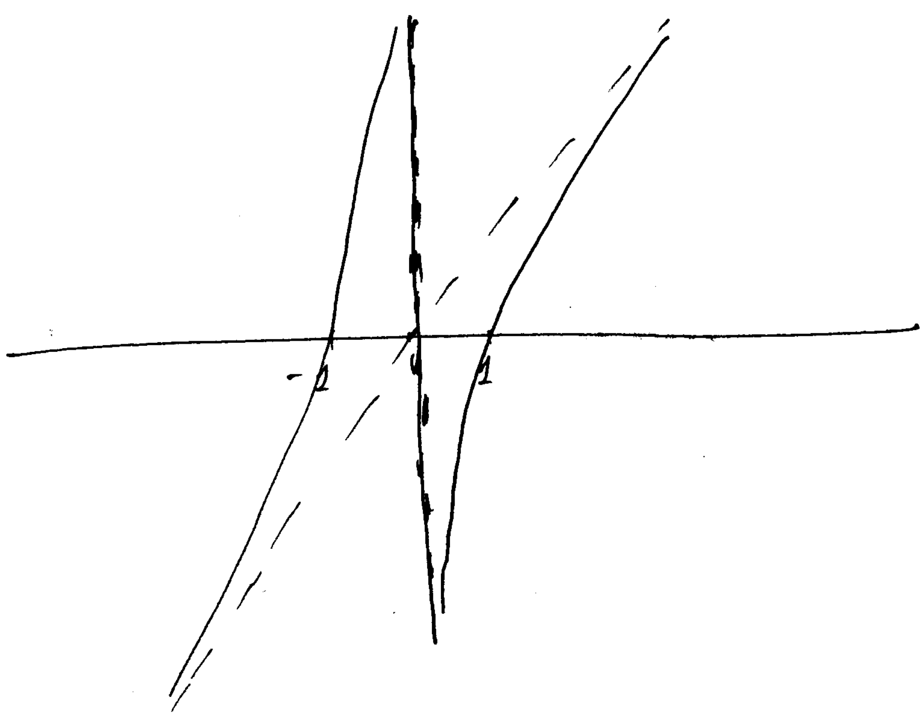
$y =$ is an oblique asymptote

(continued) (d) Find the intersepts with the coordinate axes.

$y(0)$ - is not defined

$y = 0 \Rightarrow x = \pm 1.$

(e) Use (a)–(d) to sketch the graph of the function.



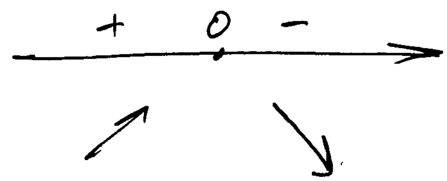
(Very) Similar to one of the graphs on the practice fina

Problem 10. Let $y = e^{-x^2/2}$.

(a) Determine where this function is increasing and decreasing; find all points of local minimum and maximum, as well as the values of the function at those points.

$y = e^{-x^2/2} \cdot (-x)$ $\qquad$ $x=0$ - local extremum.

$$\begin{cases} y' > 0 & \text{for } x < 0 \\ y' < 0 & \text{for } x < 0 \end{cases}$$



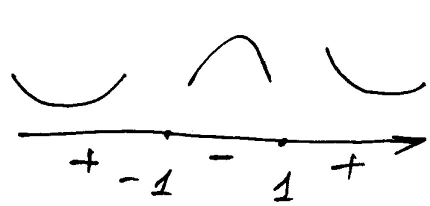
$x=0$ - local maximum

(b) Determine where the function is concave up and down and find all inflection points.

$$y'' = -\left(e^{-x^2/2} + x \cdot e^{-x^2/2} \cdot (-x)\right) =$$

$$= -e^{-x^2/2} \cdot (1 - x^2) = 0 \implies x = \pm 1.$$

$$-e^{-x^2/2} \cdot (1-x) \cdot (1+x) = 0$$



$$y(\pm 1) = e^{-1/2} = \frac{1}{\sqrt{e}}.$$

(c) Determine all asymptotes.

No vertical asymptotes.

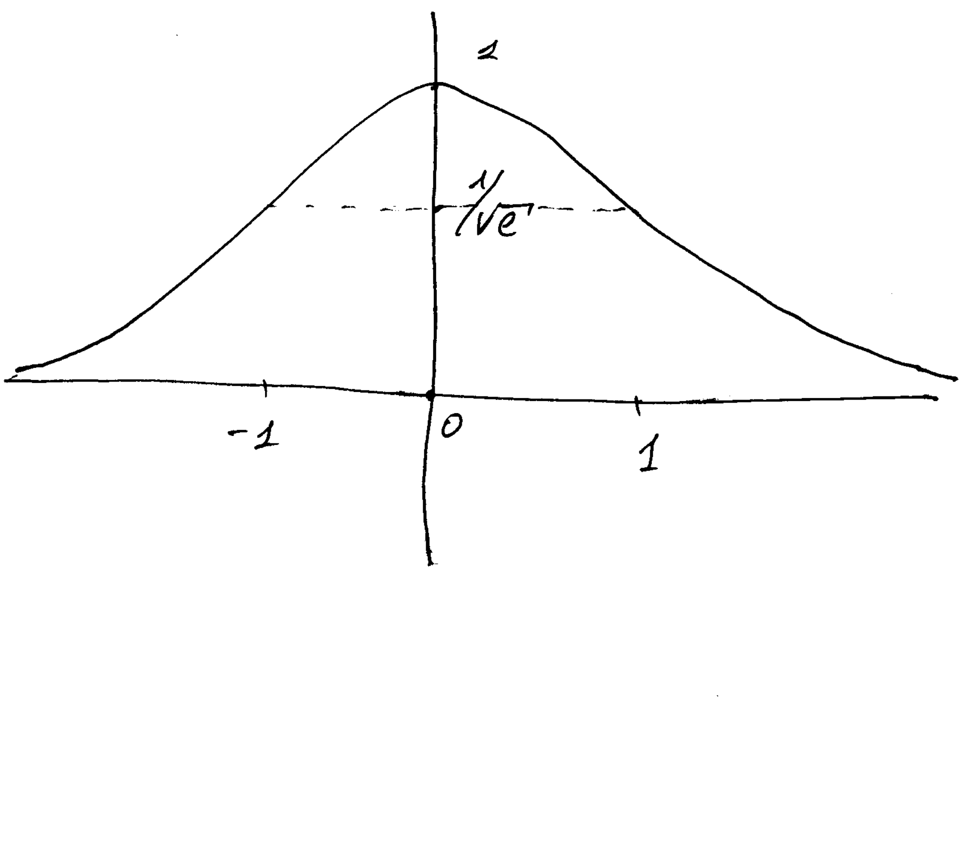
$\lim\limits_{x \to \pm\infty} e^{-x^2/2} = 0 \implies y = 0$ is a horizontal asymptote.

(continued) (d) Find the intersepts with the coordinate axes.

$y(0) = e^0 = 1.$

no intercepts with x-axis.

(e) Use (a)–(d) to sketch the graph of the function.



Problem 11. Find the limit: $\lim_{x\to 0^+} \frac{\sqrt{x}}{\ln(1+x)}$

$$\lim_{x\to 0^+} \frac{\frac{1}{2\sqrt{x}}}{\frac{1}{1+x}} = \lim_{x\to 0^+} \frac{x+1}{2\sqrt{x}} = \lim_{x\to 0^+} \left(\frac{\sqrt{x}}{2} + \frac{1}{2\sqrt{x}}\right) =$$

$$= +\infty.$$

Problem 11 is similar to problems on L'Hospital's Rule from section 5.5.1 (p. 307) (hw 10) (e.g. 1, 5, 6, 7, 12)

Problem 12. Find the limit $\lim_{x\to 0}\left(1+\frac{4}{x^2}\right)^{2x^2}$

$$= \lim_{x\to 0} e^{\ln\left(1+\frac{4}{x^2}\right)^{2x^2}} = \lim_{x\to 0} e^{2x^2\cdot \ln\left(1+\frac{4}{x^2}\right)} =$$

$$= \lim_{x\to 0} e^{\frac{\ln\left(1+\frac{4}{x^2}\right)}{1/2x^2}}$$

$$\lim_{x\to 0}\frac{\ln\left(1+\frac{4}{x^2}\right)}{1/2x^2} = \lim_{x\to 0}\frac{1}{1+\frac{4}{x^2}}\cdot\frac{8/x^3}{2/2x^3} =$$

$$= \lim_{x\to 0}\frac{4}{1+4/x^2} = \lim_{x\to 0}\frac{4x^2}{4+x^2} \stackrel{!}{=} 0.$$

$$\text{So, } \lim_{x\to 0}\left(1+\frac{4}{x^2}\right)^{2x^2} = e^0 = 1.$$

Similar to # 36, 39 (p. 307)
hw 10.