MATH 32B
SECOND MIDTERM EXAMINATION

Solutions

Nov. 20th, 2006

Please show your work. You will receive little or no credit for a correct answer to a problem which is not accompanied by sufficient explanations. If you have a question about any particular problem, please raise your hand and one of the proctors will come and talk to you. At the completion of the exam, please hand the exam booklet to your TA. If you have any questions about the grading of the exam, please see the instructor *within 15 calendar days of the examination.*

Name:____________________ Section:____________________

#1	#2	#3	#4	#5		Total

Problem 1. Sketch the vector field $F(x, y) = (x - y)i + xj$ on the plane. (Here i is the unit vector along the x-axis and j is the unit vector along the y-axis).

$F(x,y)\big|_{y=0} = xi + xj$

$F(x,y)\big|_{x=0} = -yi$

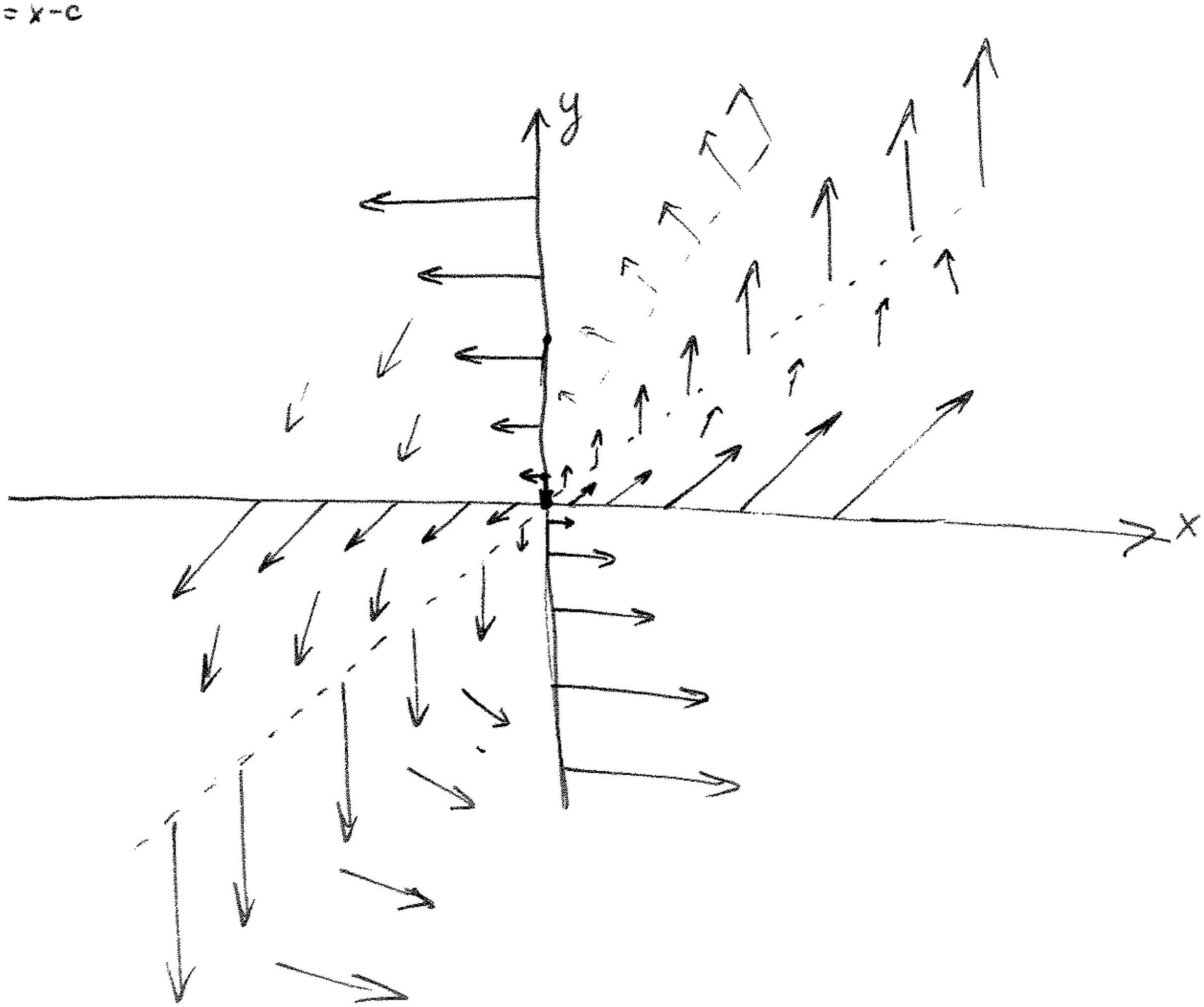
$F(x,y)\big|_{x=y} = xj$

$F(x,y)\big|_{x-y=c} = ci + xj$

constant

$x - y = c \Rightarrow y = x - c$

$F(x,y)\big|_{y=x-c} = ci + xj$



(2). $\vec{F} = yz\vec{i} + xz\vec{j} + (xy+2z)\vec{k}$

Let $f(x,y,z)$ be such that $\vec{F} = \nabla f$.

Then
$$\begin{cases} \dfrac{\partial f}{\partial x} = yz & \Rightarrow \quad f = xyz + g(y,z) \\ \dfrac{\partial f}{\partial y} = xz & \Rightarrow \quad f = xyz + h(x,z) \\ \dfrac{\partial f}{\partial z} = xy + 2z & \quad f = xyz + z^2 + k(x,y) \end{cases}$$

$$g(y,z) = h(x,z) = z^2 + \underbrace{k(x,y)}_{0}$$

$\Downarrow$

both functions do not depend on x, y.

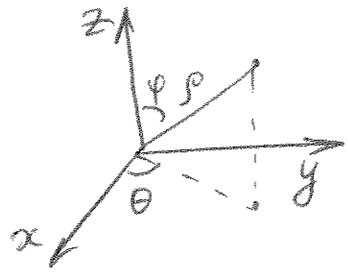
$$\Rightarrow \boxed{f = xyz + z^2}$$

(b) $(1,0,-2)$; $(4,6,3)$

$$\int_C \vec{F}\, d\vec{r} = \int_C \vec{\nabla f}\, d\vec{r} = f(4,6,3) - f(1,0,-2) =$$

$$= (72 + 9) - (4) = 77.$$

3.



$$I = \iiint_E f(x,y,z)\, dV$$

$$E = \left\{ (x,y,z) \;\middle|\; \begin{array}{l} x^2+y^2+z^2 \le a^2 \\ x^2+y^2 \le z^2 \end{array} \right\} =$$

$$= \left\{ (\rho,\theta,\varphi) \;\middle|\; \begin{array}{l} \rho^2 \le a^2 \\ \rho^2 \sin^2\varphi \le \rho^2 \cos^2\varphi \end{array} \right\} =$$

$$= \left\{ (\rho,\theta,\varphi) \;\middle|\; \rho^2 \le a^2 ; \quad \tan^2\varphi \le 1 \right\}$$

$$\left. \begin{array}{l} \tan^2\varphi \le 1 \\ 0 \le \varphi \le \pi \end{array} \right\} \iff \begin{array}{c} 0 \le \varphi \le \pi/4 \\ \text{and} \\ \frac{3\pi}{4} \le \varphi \le \pi . \end{array}$$

Thus, $$E = \left\{ (x,y,z) \;\middle|\; \rho^2 \le a^2 , \quad \begin{array}{c} 0 \le \varphi \le \pi/4 \\ \text{and} \\ \frac{3\pi}{4} \le \varphi \le \pi \end{array} \right\}.$$

Jacobian $\dfrac{\partial (x,y,z)}{\partial (\rho,\theta,\varphi)} = \rho^2 \sin\varphi .$

$$I = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{a} f \cdot \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta + \int_0^{2\pi}\int_{3\pi/4}^{\pi}\int_0^{a} f \cdot \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

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(4).

$$I = \int_C xy\,dx$$

$C: \quad y = \sin x, \quad 0 \le x \le \pi/2 \qquad \Big| \qquad x = t, \quad y = \sin t, \quad 0 \le t \le \pi/2.$

$$I = \int_0^{\pi/2} t \sin t\,dt = -t\cos t \Big|_0^{\pi/2} + \int_0^{\pi/2} \cos t\,dt = \sin t \Big|_0^{\pi/2} = 1$$

$u = t \qquad \Big| \qquad du = dt \qquad \Big| \quad \cancel{\pi(0)}$

$dv = \sin t\,dt \qquad \Big| \qquad v = -\cos t \qquad \Big|$

5).

Green's formula:

$$\int_C P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

with $P = -y$, $Q = x$

implies

$$\frac{1}{2} \int_C x\,dy - y\,dx = \frac{1}{2} \iint_S 2\,dx\,dy = \iint_S dx\,dy = A.$$