Introduction to Vassiliev invariants.

First Idea: think of a crossing change as of a continuous process.
Critical moment – when 2 strands intersect.
At this moment, the object we have is not an ordinary knot.

Singular knots & double pts: Extend the class of objects we consider to include singular knots, i.e., smooth (or continuous) maps
$k: S^1 \to \mathbb{R}^3$ with no singularities except a finite number of transversal self-intersections (double pts of k).

Relation between invariants of ordinary and singular knots:

$\mathcal{V}$ – ordinary knot invariant
$\mathcal{V}^{(m)}$ – invariant of singular knots with exactly m double points.

$$\mathcal{V}^{(0)} = \mathcal{V}$$

Vassiliev invariants ←

$\mathcal{V}^{(m)}($  $) = \mathcal{V}^{(m-1)}($  $) - \mathcal{V}^{(m-1)}($  $)$

double pt
(we will circle them)

Anology with derivatives in calculus:

$\mathcal{V}^{(m)}$ ~ m'th partial derivative (of a knot invariant V)

$\mathcal{V}^{(m-1)}$ ~ $(m-1)^{th}$ partial derivative

The formula above is: derivatives as "differences".

if $v^{(m+1)} \equiv 0$ (for all singular knots with $(m+1)$ double pts).

(The main relation then implies that $V^{(n)} \equiv 0$ $\forall n \geq m+1$).

(This is recorded as $v(\underbrace{\times \times \ldots \times}_{m+1}) = 0$

<u>Remark</u>. Assuming Vassiliev invariants exist, they don't have to be unique (e.g., can multiply

<u>Questions</u>: 1. Which known knot invariants are Vassiliev invariants?

2. Do Vassiliev invariants classify knots?

3. Is there an analog of Taylor's formula (i.e., can an arbitrary knot invariant be approximated by Vassiliev invariants?)

<u>Simple properties.</u>

<u>Lemma 1.</u> $v($  $) = 0$ (~~[illegible]~~).

<u>Proof</u>

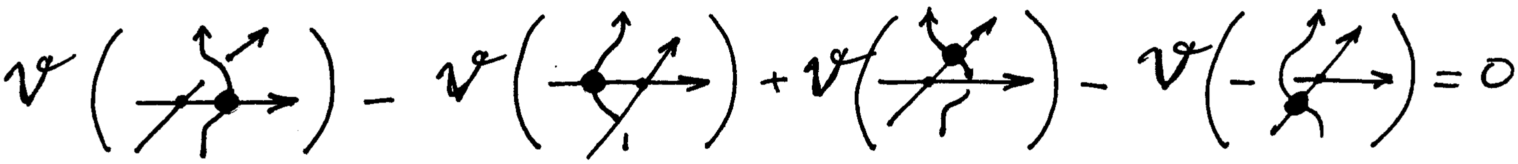
$v($  $) = v($  $) - v($  $) = 0$,

since the two knots on the right hand side are equiv. ∎

More generally, for any knot of the form

$K =$  , $v(K) = 0$. (1-term relation)

Lemma 2.



Proof. Use the double points indicate above ~~in~~ in the main relation of Vassiliev invariants. (Notice that we always keep the same double pt untouched).
Then we get the sum of the form

$$(a-b)-(c-d)+(c-a)-(d-b)=0.$$

■

Vector space structure

V^m – inv. of order m. $\quad V=\cup V^m$

$V^0\subset V^1\subset V^2\subset V^3\subset\dots$

If $v, w\in V$, then $u = \alpha v+\beta w$ satisfies the same linear relation.

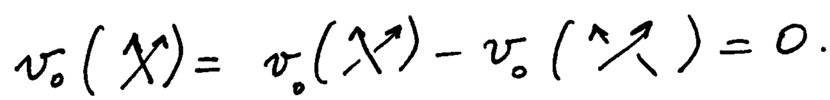
Thus, V is a vector space (inf.-dim.).
It is filtered by the sequence of inclusions

$$V_0\subset V_1\subset V_2\subset\dots$$

Invariants of order 0

$v_0\in V^0$ vanishes on any knot with 1 double pt.

Thus,



In particular, v_0 is invariant under a crossing change.

Since by a series of crossing changes any knot can be modified to become the unknot, v_0 has the same value on all ordinary knots. $\boxed{V^0=\mathbb{R}}$

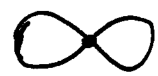
2 double pts.

<u>Lemma.</u> v_1 vanishes on a singular knot with just 1 double pt.

<u>Proof.</u> Let K_1 be a knot with one double pt.
Let K_1' be obtained from K by 1 crossing change.

Then $$v_1(K_1) - v_1(K_1') = v_1(K_2) = 0.$$
where K_2 has 2 double pts.

Thus, $v_1(K_1) = v_1(K_1')$.

Thus, v_1 has the same value on all the knots in V^1.
By crossing changes, any knot in V^1 can be modified to .

All Vassiliev invariants on this knot vanish by 1-term relation. Thus, $V^1 = V^0 = \mathbb{R}$.

More generally,

<u>Lemma.</u> The value of an invariant v_n of order n on a knot with exactly n double pts does not vary under crossing changes in the knot.

<u>Proof</u> Let K_n be of order n (has n double pts).

$$v_n(K_n) - v_n(K_n') = v_n(K_{n+1}) = 0.$$

Thus, $v_n(K_n)$ is the same as

v_n ().

n double pts

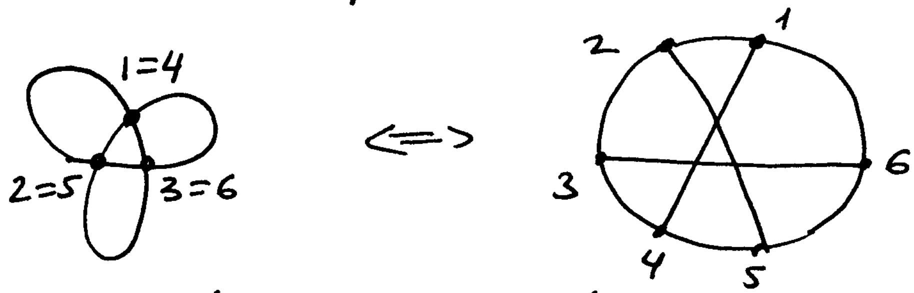
of $v_n \in V^n$ on K_n depends only on the sequence in which double pts appear (and does not depend on knotting).

Let $K: S^1 \to \mathbb{R}^3$ be a knot with n double pts.

~~Moving around a circle~~

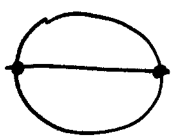
<u>Gauss diagram:</u> - oriented circle;
- marked finitely many chords,
(up to orientation-preserving diffeos)

<u>Ex.:</u>

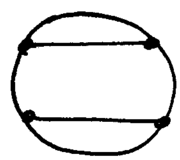


Gauss diagrams of different orders: (order = # (double pts))

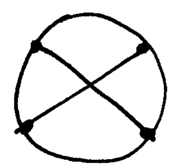
<u>n=1</u> :



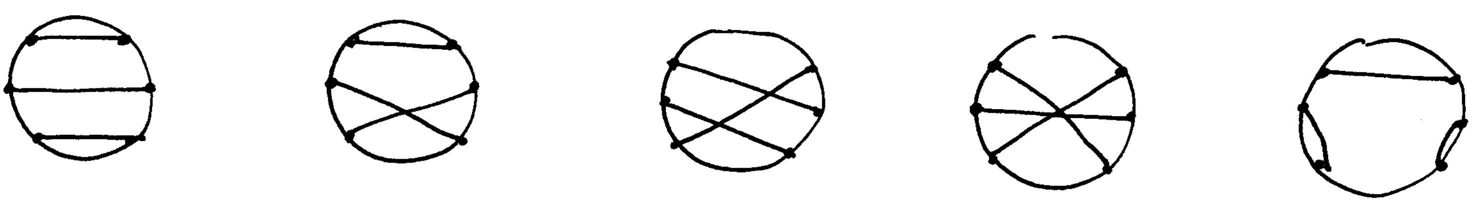
n = 2 :



&

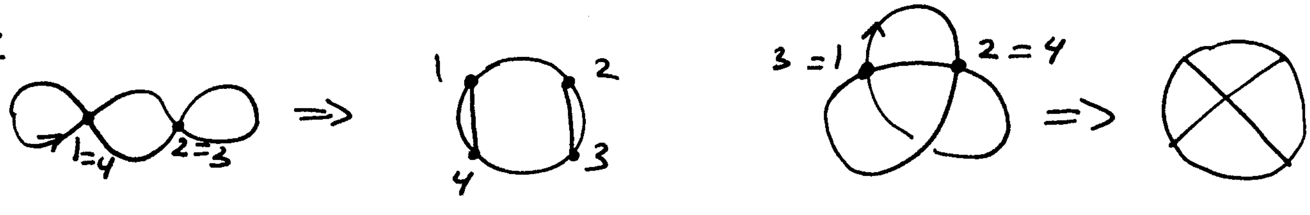


<u>n=3</u> :



<u>Lemma.</u> Any Gauss diagram is a diagram of some singular knot.

<u>Ex.</u>

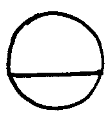
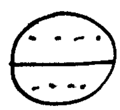


V_n/V_{n-1} – inv. of order n up to inv. of order (n-1).

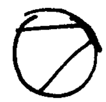
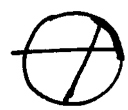
Symbol of v_n = Restriction of v_n to knots with exactly n double pts.

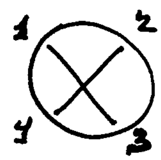
Lemma.
The value of Symbol on a ~~diagram~~ knot with n double pts depends only on the Gauss diagram of the knot.

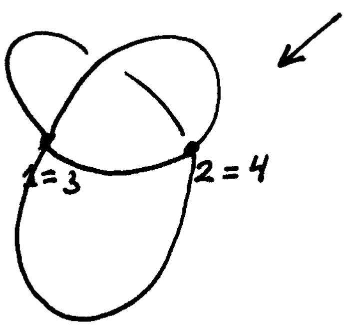
One-term relations on the language of diagrams:

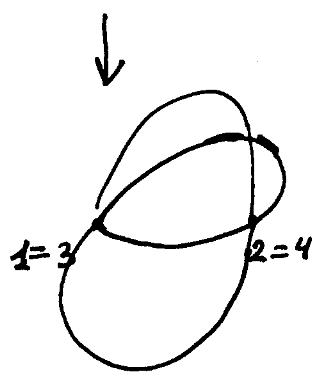
 $= 0$;  $= 0$.

4-term relation:

 $-$  $+$  $-$  $= 0$.

E.g., let $D =$  be a Gauss diagram.





– The two diagrams are related by a crossing change. Thus, the value of $v_2 \in V^2$ on these diagrams are defined and equal.

More generally,

<u>Thm.</u> Vassiliev invariants of type m (i.e., V^m) $\Longleftrightarrow$ Functions on Gauss diagrams with 2m points satisfying properties analogous to 1-term and 4-term relations:

the function on such a diagram is 0. $= 0$

(2). (see above, 4-term relation).

Examples.

(1). Coefficients of the Conway polynomial.

Recall the skein relation:

$$C(L_+) - C(L_-) = z \cdot C(L_0)$$

Then

$$C(L_\times) = C(L_+) - C(L_-) = z \cdot C(L_0)$$

If K has 1 double pt, $C(K)$ is divisable by z.
Inductively, if K ~~is divisable by~~ has $(m+1)$ double points, $C(K)$ is divisable by z^{m+1}. Thus, the coefficient of z^m in $C(K)$ is 0. Thus, $(m+1)^{th}$ coefficient of $C(K)$ is a Vassiliev invariant of type $(m+1)$

(2). Coefficients of the HOMFLY polynomial.

Skein relation

$$a\,P(L_+) - a^{-1} P(L_-) = z\,P(L_0).$$

Another parametrization:

$$z = \sqrt{q} - \frac{1}{\sqrt{q}}, \quad a = q^{N/2}.$$

Get

$$q^{N/2} P(L_+) - q^{-N/2} P(L_-) = \left(\sqrt{q} - \frac{1}{\sqrt{q}}\right) P(L_0).$$

Substitute $q = e^x$ and expand in powers of x.
The above equation:

$$P(L_+) - P(L_-) = x \cdot (\text{something}).$$

Similarly to ex.(1), m^{th} coefficient is inv. of type m.

Symbol of $v_2 \in V^2$: values on $\otimes$ and $\ominus$ are needed.

Let Δ_n be the the space of Gauss diagrams with $2n$ pts. all inv. are trivial

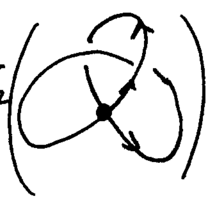
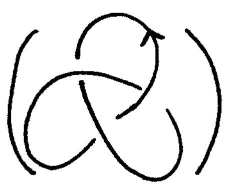
$\Delta_n = \text{span}\{\otimes\}$.

Fix a basis invariant in V^2 as:

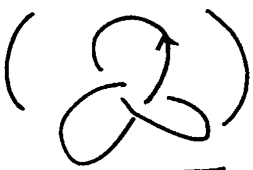
$$v_2(\otimes) = 1, \quad v_2(\bigcirc) = 0.$$

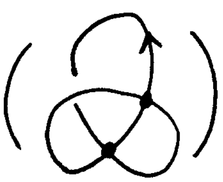
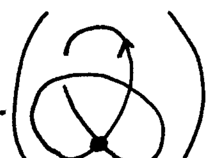
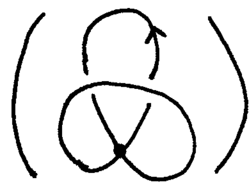
<u>Trefoils</u>

 (right trefoil)

$v_2\Big($  $\Big) = v_2\Big($  $\Big) - \underbrace{v_2\Big(}_{}$  $\Big)$ $=$ 0

unknot

$= v_2\Big($  $\Big)$

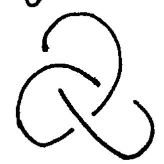
$v_2\Big($  $\Big) = v_2\Big($  $\Big) - v_2\Big($  $\Big)$ $= 0$

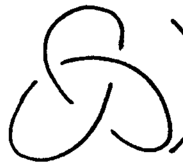
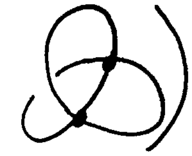
Thus, $v_2\Big($  $\Big) = v_2\Big($  $\Big)$

Similarly, $v_2\Big($  $\Big)$ //

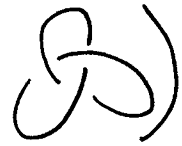
Thus, v_2 does not distinguish the right and left trefoils.

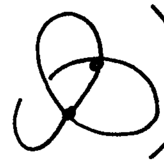
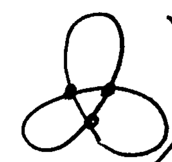
Similarly to order 2 invariants, we

v_3 () = v_3 ()

v_3 () = v_3 ()

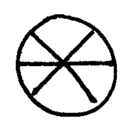
Thus,

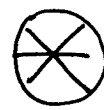
v_3 () − v_3 () =

v_3 () − v_3 () = v_3 ()

⇕

Gauss diagram

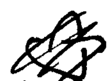


If v_3 () = 1,

then the values of v_3 on the left and right trefoils differ by 1.

Thus, v_3 distinguishes the trefoils.

<u>Homework</u>: 1. Let $v_2 \in V^2$ be an invariant of order 2 normalized so that $v_2(\otimes) = 1$, $v_2(O) = 0$. Compute $v_2(4_1)$ (the value of this inv. of figure 8 knot).

2. Compute v_2 on the torus knot (2,5). (You may use the diagram   for this knot).

<u>Kontsevich's theorem.</u> 1). Vassiliev invariants of order n exist.

2). The quotient space V_n/V_{n-1} is isomorphic to the space $\Delta_n^* / \{1\text{-term rel.}; 4\text{-term rel.}\}$ (i.e. the space of linear functions Δ_n^* on Δ_n modulo the 1-term & 4-term relations).