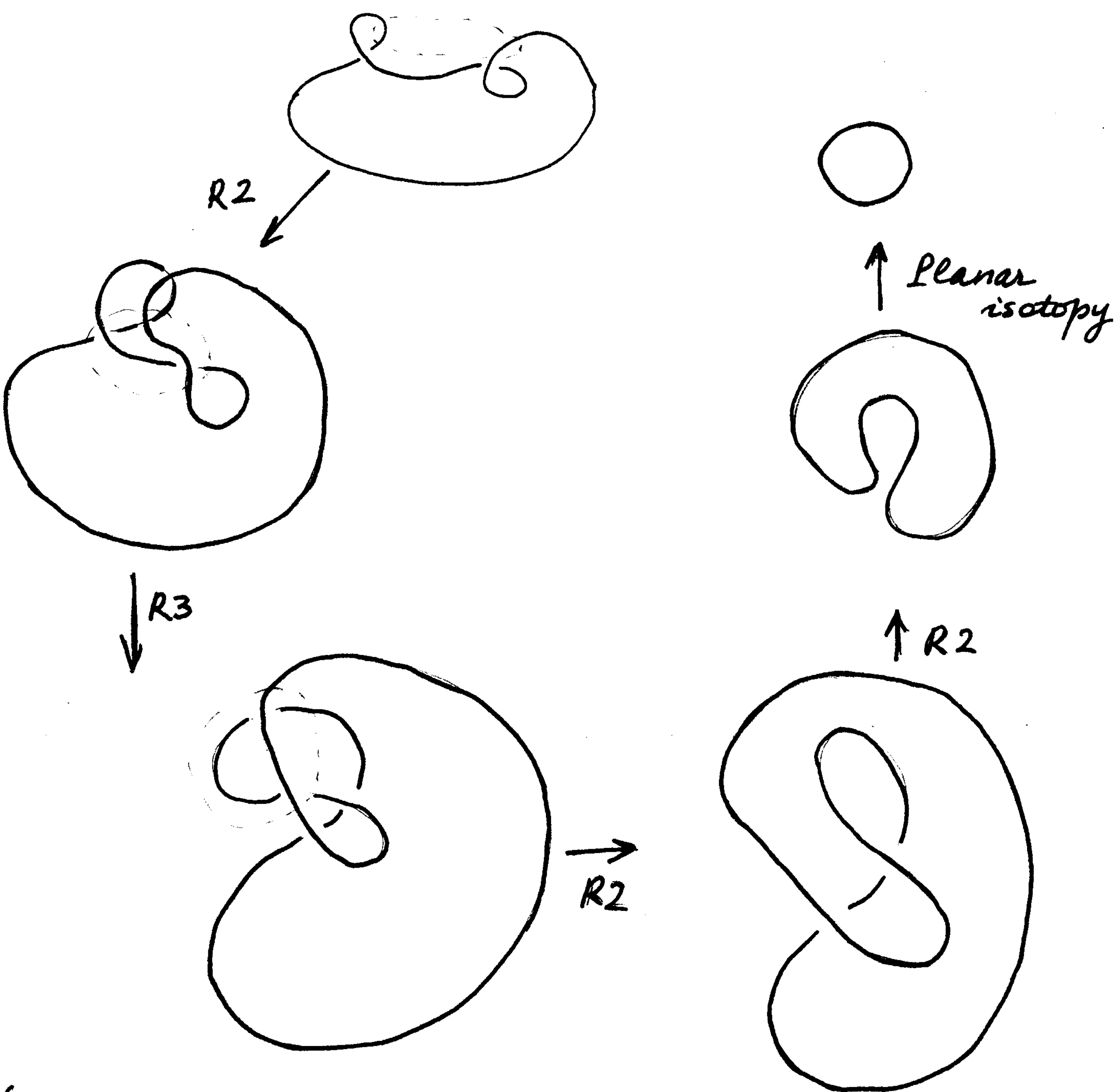
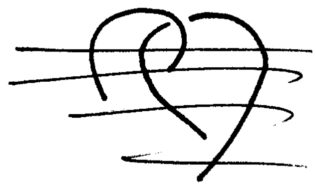
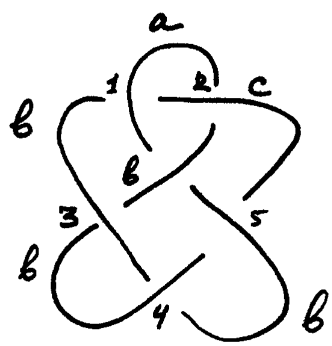
Problem 1. Show that there is a sequence of Reidemeister moves R2 and R3 transforming the diagram below into the standard diagram of the unknot.



(On each diagram, the fragment circled by ⁝ is modified by the move indicated over the next arrow).

Problem 2. Compute the 3-coloring invariant of the knot 6_2.





1). Let a, b, c be the colors of the three arcs meeting at the crossing 1.

2). Assume $a \neq b \neq c$. Then at crossing 2 we can determine the color of the remaining arc as b.

3). At crossing 3, all the arcs have color b.

4). Hence, same at crossing 4

5). Crossing 5: $c = b$

6). Crossing 1 or 2: $a = c$.

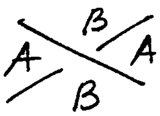
Thus, $a = b = c$. $\Rightarrow$ There are only 3 (trivial) 3-colorings.

(Another method is to set up the 6×6 matrix describing the crossings (which arcs meet at each crossing), then findings its raw reduced echelon form and determining nullity of the matrix. The number of 3-colorings is then 3^{nullity}).

Problem 3. Define the bracket polynomial of a link K as the polynomial in A, B, d such that

$$\langle K \rangle = \Sigma_\sigma \langle K|\sigma\rangle \cdot d^{||\sigma||},$$

where $\langle K|\sigma\rangle$ is the product of labels attached to the state σ.



1. Prove the relations:
(a) $\langle K \sqcup O \rangle = d \cdot \langle K \rangle$;
(b) $\langle \times \rangle = A \cdot \langle \asymp \rangle + B \cdot \langle \supset\subset \rangle$
2. Using these relations, prove that

$$\langle \text{⟳} \rangle = A \cdot B \langle \supset\subset \rangle + (ABd + A^2 + B^2) \langle \asymp \rangle$$

See lecture notes.

Problem 4. Show (by any valid method) that 4_1 is ambient isotopic to its mirror image.

Method 1:

Find Reidemeister moves from 4_1 to $\overline{4_1}$.
(see hw 1).

or

Method 2:

Compute the bracket polynomial (e.g., by using state sums, or the recurrence relation) and show that it is invariant under the change $A \leftrightarrow A^{-1}$!

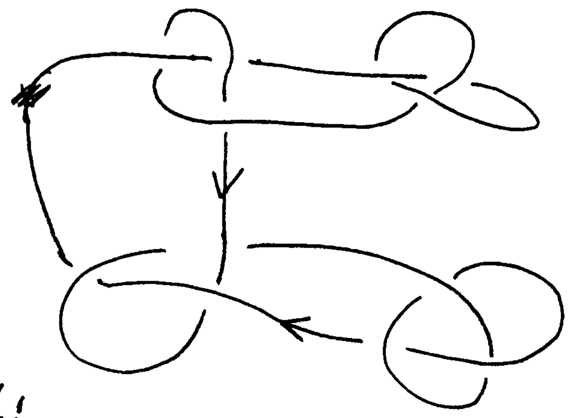
or

Method 3:

Compute the Jones polynomial (using the skein relation) & show its invariance under $t \leftrightarrow t^{-1}$!

Problem 5. 1. What is the Jones polynomial of the connected sum of two knots? (Find $V_{K_1 \# K_2}(t)$ in terms of $V_{K_1}(t)$ and $V_{K_2}(t)$.)

2. Let T be the right trefoil. Given that $V_T(t) = t - t^3 - t^4$, compute the Jones polynomial of the following knot:

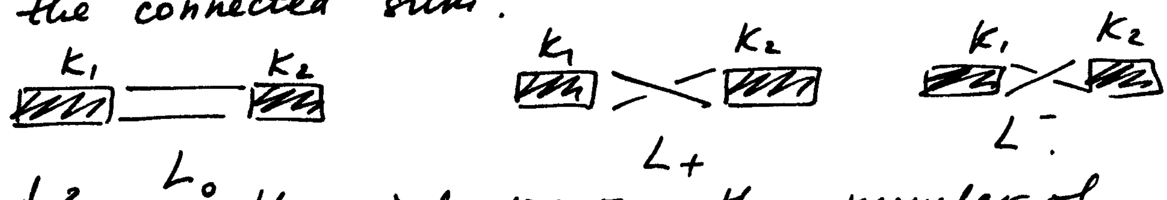


Method 1:

1. It is enough to prove that $\langle K_1 \# K_2 \rangle = \langle K_1 \rangle \cdot \langle K_2 \rangle$ (This implies that $V_{K_1 \# K_2} = V_{K_1} \cdot V_{K_2}$).
To show the statement for $\langle\ \rangle$, represent each of the states of $K_1 \# K_2$ in termes of the states of K_1 & K_2.

Method 2:

Use the skein relation of the Jones polynomial applied to the crossing "at the place where you take the connected sum":



Method 3: Use induction on the number of crossings of K_2.

2. The given knot is the connected sum of 4 trefoils. Thus, $V_K = (t - t^3 - t^4)^4$

Problem 6. Compute (inductively) the Jones polynomial of the n-component unlink.

2 components:

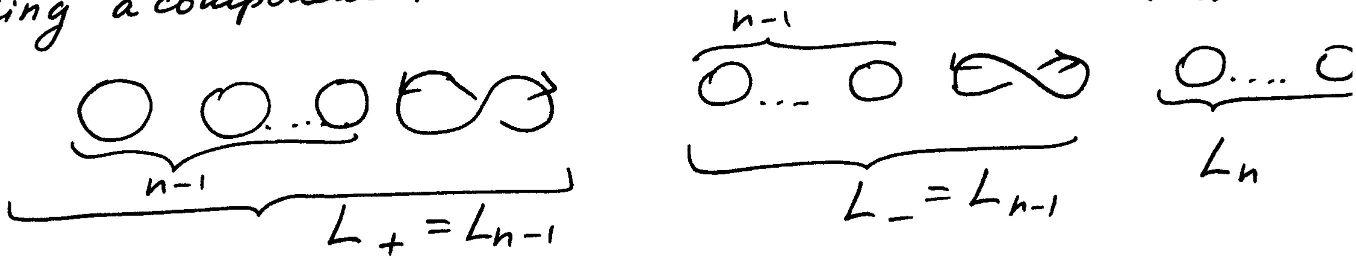
$$t^{-1} - t = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_{L_2}$$

$$V_{(L_2)} = \left(-\sqrt{t} - \frac{1}{\sqrt{t}}\right) = -\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right) =$$

$$= -\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right) \cdot V_{(L_1)}$$

Induction:

adding a component:



$$t^{-1} V_{L_{n-1}} - t V_{L_{n-1}} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) V_{L_n}$$

$$V_{L_n} = - V_{L_{n-1}} \cdot \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)$$

Thus, $\boxed{V_{L_n} = (-1)^{n-1} \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^{n-1}}$.