USING BRAIDS TO UNRAVEL KNOTS

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Abstract. The central goal of this paper is to discuss how braid groups and representations of these braid groups can be used to describe and implement an algorithm to search relatively efficiently for nontrivial knots with trivial Jones polynomial. The question of whether such a knot exists is still open. The point of view of the braid group taken here is topological rather than combinatorial. After suitable topological preliminaries, we define the main items used in all of our constructions and outline our basic strategy. As a complete and illustrative example, we prove that the Burau representation of the braid group B_3 is faithful (injective). Finally, we describe how the statements of the lemmas used to prove this faithfulness carry over to dealing with the braid group B_4.

Topological Definitions

For the purposes of what follows, we'll need a more topological definition of a braid. Then we can use topological ideas to prove theorems.

Definition. A *topological space* is a set X together with a collection S of subsets of X such that

(1) For any subsets $U, V \in A$, $U \cap V \in A$.
(2) For any collection $\{U_i\}_{i\in I}$ of subsets $U_i \in A$, the union of the U_i is also in A.
(3) $\emptyset, X \in A$.

The elements of X are called *points* and the elements of A are called *open sets*.

Basically, a topological space is a set where we can define all the usual notions of continuity, convergence etc.

Definition. Let X, Y be topological spaces. A function $f : X \to Y$ is called a *homeomorphism* if it is invertible and both f and f^{-1} are continuous. In this case, X is said to be "homeomorphic to Y via f".

Definition. Let D be the unit disk in $\mathbb{R}^2$ centered at the origin. Choose n *puncture points* $P = \{p_1, \dots, p_n\}$ along the diameter such that $-1 < x_1 < \cdots < x_n < 1$ where x_i is the x-coordinate of p_i, and define $D_n = D - P$. The *braid group* B_n is the set of equivalence classes of homeomorphisms $h : D \to D$ such that $h(P) = P$ where two homeomorphisms h_1, h_2 are equivalent if there is a homeomorphism $\alpha : D \to D$ isotopic (relative to the boundary $\partial D_n = \mathbb{S}^1 \cup P$) to the identity such that $h_1 = h_2 \circ \alpha$.

One can think of this in terms of the old definition by considering that the strands start at the points p_i and twist around each other before connecting again, possibly in a different order to an identical disk above it. A single braid is really just a fixed twisting of the strands, which can be achieved by deforming one of the disks to mix the strands up, for instance, by twisting a small part of the disk so that two of the points are switched:

(SEE strandtwistdiagram.bmp)

In fact, the $n-1$ maps σ_i that accomplish this (σ_i switches p_i with p_{i+1} and leaves the rest alone) generate the whole set of homeomorphisms. Here is an explicit definition of one such σ_i

Example. Let $R(\theta)$ be the 2×2 rotation matrix about the origin by θ radians. Let $r = r(x, y)$ be the distance from the origin and fix two points p_i, p_j in the disk on the x-axis, both distance $d > 0$ from the endpoints of the diameter going through them. Define $\sigma : D \to D$ by

$$\sigma(x,y) = \begin{cases} R\big(\big(\frac{r+d-1}{d}\big)\,\pi\big)\cdot(x,y) & \text{if } r \geq d; \\ R(\pi)\cdot(x,y) & \text{if } r < d \end{cases}$$

This function σ rotates the disk so that there is no rotation on the boundary, full rotation by π of the disk whose boundary contains p_i, p_j and a smooth transition of rotation from 0 to π as the points move closer to the center. This map is clearly continuous and invertible with a continuous inverse, i.e. a homeomorphism. Note that we allow homeomorphisms $h \in B_n$ to move the puncture points around, while the isotopy we use to determine whether h is to be ignored cannot do this, so this really is just like the ambient isotopy we have already seen.

This is actually a special case of what is called a *Dehn twist*, which we will use later. A Dehn twist about a simple closed curve is accomplished as follows. A simple closed curve C is homeomorphic to the unit circle $\mathbb{S}$ via some $h : C \to \mathbb{S}$. We apply this homeomorphism so that we may work with the unit circle $\mathbb{S}$. We then apply a twisting map similar to the above map σ except that this map rotates every point in the annulus $D_2^2 - D_1^2$ (D_k^n is the n-dimensional disk of radius k) by some angle $\theta(r)$ which is a smooth function of r. The initial angle is fixed at $\theta(1) = 0$ and the terminal angle $\theta(2)$ is usually chosen to be π or 2π but can be anything.

We will also be using the notion of a *covering space*:

Definition. Let X, Y be a topological spaces and $f : X \to Y$ a continuous map. A point $a \in Y$ is *evenly covered* (by f) if there is some neighborhood U containing a such that every connected component C of $f^{-1}(U)$ is homeomorphic to U via f.

Accordingly,

Definition. Let X be a topological space. A *covering space* is a topological space $\tilde{X}$ and a continuous function $\pi : \tilde{X} \to X$ such that every point is evenly covered by π. We will sometimes refer to such a covering space as the pair (X, π) or simply as $\tilde{X}$ when the map π is clear.

The canonical example of a covering space is the following: Let X be the unit circle $\mathbb{S} \subset \mathbb{R}^2 \subset \mathbb{R}^3$, $\tilde{X}$ be the helix in $\mathbb{R}^3$ whose axis goes through the center of the circle and is orthogonal to the plane of the circle and π be the projection onto the xy-plane. The covering space here is the space $(\tilde{X}, \pi)$. We have actually already seen an example of a covering space in class: in the proof that any knot can be unknotted by performing one or more crossing changes, we started at a point and constructed an unknot in $\mathbb{R}^3$ by tracing out the knot as we simultaneously moved upwards in the z-axis. We arrived at an unknot whose projection had the same universe as our original knot.

The main facts about covering spaces (for our uses) that we will use are

Theorem. *Let X be a topological space and $(\tilde{X}, \pi)$ its covering space. For any path $f : [0,1] \to X$ and any $\tilde{x}_0 \in \tilde{X}$ where $\pi(\tilde{x}_0) = x_0 = f(0)$, there is a path $\tilde{f} : [0,1] \to \tilde{X}$ such that $\pi \circ \tilde{f} = f$. In this case, $\tilde{f}$ is called a "lift" of f (to $\tilde{X}$ via π).*

One can show that this lifting construction descends to the homotopy equivalence classes. Moreover,

Theorem. *For any subgroup H of the fundamental group (explained on the board), there is a covering space $\tilde{X}_H$ such that the image of the fundamental group under π is isomorphic to H.*

The Main Algorithm

Definition. A *representation* of a group G on a vector space V is a group homomorphism $F : G \to \mathrm{GL}(V)$. The representation is *faithful* if F is injective, i.e. if the only element $g \in G$ such that $F(g) = \mathrm{id}_V$ is $g = 1$.

A representation of a group G, in our case the braid groups B_n, allows us to use the extremely regular structure of linear spaces in working with our group. In particular, we can take all the usual invariants, such as determinant, trace, characteristic polynomial, etc., and attach them to elements in our group in some fashion.

In a recent paper ("The Burau representation is not faithful for n=5", Stephen Bigelow, 1999), an algorithm was presented to possibly locate a non-trivial knot K having trivial Jones polynomial (i.e. $V_K(t) = 1$). The algorithm rests upon the fact that the Burau representation is not faithful for B_4.

Many others had collectively proven that the same representation is faithful for B_n where $n \leq 3$ and unfaithful for $n \geq 6$. The author used a similar algorithm to successfully prove this fact for B_5.

One may ask what is special about B_4 and why the algorithm could not already be used for any of the B_n where $n \geq 5$. The reason is the following. There are three representations in action here: the Burau representation, the Jones representation and the Temperley-Lieb representation and different representations give different invariant polynomials, for instance the Jones versus the Alexander. But some representations are easier to work with than others. In the special case of B_4, we have the following:

Proposition 1. *The following are equivalent:*

1. *The Burau representation of B_4 is faithful.*
2. *The Temperley-Lieb representation of B_4 is faithful.*
3. *The Jones representation of B_4 is faithful.*

Thus once we prove that the Burau representation B_4 is not faithful, we have access to a whole collection of braids B such that the knot $K = \overline{B}$ obtained via braid closure has Jones polynomial $V_K(t) = 1$. It is hoped that at least one of these braids will give a non-trivial knot K.

What is promising about this method is that while it is in fact just a "search engine" for knots, the algorithm will actually search through knot diagrams with a number of crossings *far* beyond anything that has been systematically done before. For instance, a computer search for two curves (these are the cornerstone of the idea; explained below) showed that any these curves must intersect each other at least 500 times, which is directly related to the writhe of the closed knot $K = \overline{B}$. The reason it is able to search this far is because the ideas used below allow for a much much much more efficient search than simply taking knots and doing special knot moves to them.

Results Used to Develop the Algorithm

As an illustrative example, we prove that the Burau representation of B_3 is faithful.

There are two objects that we will use in this proof.

Definition. Fix $d_0 = (0, -1)$ on D_n. A *fork* F is a union of three curve segments e_1, e_2, e_3 and four vertices d_0, p_i, p_j, z with $z \in D_n$ such that

1. F intersects only the puncture points p_i, p_j.
2. F intersects the boundary of the disk only at d_0.
3. e_1, e_2, e_3 all have z as a vertex.

The union of the segments e_2, e_3 not having d_0 as a vertex is called the *prong* $P(F)$ of F.

A *chord* is a curve segment having vertices d_0, w such that

1. C does not intersect the boundary of the disk except at d_0, w.
2. One of the two pieces of D_n separated by C contains exactly one puncture point.

Depiction:

(SEE fork.bmp and chord.bmp)

One object that is going to be used in this paper is a certain formal sum (i.e. a sum which does not necessarily make sense in the context of usual arithmetic). These formal sums properly belong to an area of topology called *homology.* To go into this area would take us too far outside the subject of this paper so I will not be covering it. Suffice to say it is closely related to the set of paths in the covering space $\tilde{D}_n$ of D_n. For a fork F and a chord C, we define a formal sum $\langle F, C \rangle$ by defining

$$\langle F, C \rangle = \sum_{i=1}^{k} \epsilon_i q^{a_i}$$

where $z_1, \ldots, z_k$ are the points of intersection between F and C, ϵ_i is the sign of the intersection at z_i, and a_i is the winding number of the curve γ_i that starts at d_0, goes to z_i along F then travels back to d_0 along C. The winding number is simply the number of revolutions the curve makes around each of the puncture points.

(SEE gammawinding.bmp)

Lemma 2. *Let $h : D_n \to D_n$ be a homeomorphism (i.e. representative of a braid by our above definition) in the kernel of the Burau representation. Then $\langle F, C\rangle = \langle h(F), C\rangle$ for any fork F and chord C.*

Proof. First, we construct a covering space (mentioned above) $\tilde{D_n}$ for D_n.

First, take a copy D_n^k of D_n for each $k \in \mathbb{Z}$. Then on each copy, mark the vertical line segments going from the puncture points p_i to the top of the disk. Finally, for each k identify the corresponding line segments so that curves coming from the left of the segment on D_n^{k+1} end up on the right side of the segment on D_n^k:

(SEE coveringspace.bmp)

We project onto D_n using $\pi : \tilde{D_n} \to D_n$ which takes each point $x \in D_n^k$ to the corresponding point in D_n. Using the first theorem above, we have lifts $\tilde{C}$ and $\tilde{F}$ of C and F respectively. The symbol q^a for $a \in \mathbb{Z}$ is the homeomorphism obtained by composing q (or q^{-1} if $a < 0$) with itself $\mid a \mid$ times.

Using the covering space, we can find an alternate expression for $\langle F, C\rangle$. Let $\mathrm{Int}(\alpha, \beta)$ be the intersection number (the crossing number of α and β at an intersection where α is considered to be "over" β and the direction is the direction induced by the curve parameter $t \in [0, 1]$) between two curves α, β and consider the sum

$$\sum_{a\in\mathbb{Z}} \mathrm{Int}(q^a(\tilde{C}), \tilde{F})q^a$$

Here, we are taking the homeomorphism q^a, applying it to the lifted curve $\tilde{C}$, taking the intersection number of it with the lift $\tilde{F}$ and using this number as the coefficient of q^a in the formal sum. Suppose that γ_i has winding number a_i around p_1 in the original sum. In the covering space, $\tilde{\gamma_i}$ ascends or descends by a_i sheets (depending on sign) by the geometry of the covering space. Then for $a = a_i$ in the new sum, we have translated the chord C by $a = a_i$ sheets to $q^a(C)$ so the intersection point is preserved. To generalize this, if γ_i winds around multiple points $p_1, \ldots, p_m$ then in the covering space, $\tilde{\gamma_i}$ ascends or descends by m sheets, so preservation of the intersection number is achieved by translating the chord C an equal number of times. To further generalize, if there are multiple a_i taking on the same value, then this will be accounted for since the translated chord $q^a(C)$ will intersect all of the appropriate $\tilde{\gamma_i}$, i.e. we will have a coefficient of some k instead of a sum of 1 k-times. Thus the two expressions are equal.

(did this in class; alternatively, see figure8.bmp)

There are a lot of things we have to name so I've left these in a separate diagram:

(SEE figure8.bmp)

Finally, we can define the "figure-8" $L(t) = \gamma(\delta_j(\gamma^{-1}({\delta_i}^{-1}(t))))$. When we lift this to $\tilde{D_n}$, it becomes a loop $\tilde{L}(t)$, and outside of some neighborhoods around p_i and p_j it is simply the union of $\tilde{P}$ and $q(\tilde{p})$ with the orientation reversed:

(SEE figure8.bmp)

Algebraically, outside of these neighborhoods, $\tilde{L} = (1 - q)\tilde{P}$ so

$$\langle F, C\rangle = \frac{1}{1-q}\sum_{a\in\mathbb{Z}} \mathrm{Int}(q^a\tilde{C}, \tilde{L})q^a$$

But $\tilde{D_n}$ was taken to be the covering space where the loops project onto the kernel of the Burau representation, so the loops $h(\tilde{L})$ and $\tilde{L}$ are equivalent in $H_1(\tilde{D_n}$. Therefore,

$$\langle F, C\rangle = \frac{1}{1-q}\sum_{a\in\mathbb{Z}} \mathrm{Int}(q^a\tilde{C}, \tilde{L})q^a = \frac{1}{1-q}\sum_{a\in\mathbb{Z}} \mathrm{Int}(q^a\tilde{C}, h(\tilde{L}))q^a = \langle h(F), C\rangle$$

□

The preceding lemma will be used with the next lemma to prove the main result.

Lemma 3. *For $n = 3$, $\langle F, C\rangle = 0$ if and only if $P(F)$ is isotopic to an arc disjoint from C.*

The lemmas will be used together as follows. We take any homeomorphism (read: braid) h in the kernel of the representation and construct an F and C such that $\langle F, C\rangle = 0$, note by the first lemma that $\langle h(F), C\rangle = 0$, and use the second lemma that is disjoint from C.

Proof. Let k be the number of intersection points of F and C and apply an isotopy so that k is minimal. Let these points be $z_1, \ldots, z_k$. The direct statement is obvious: when $k = 0$, the sum is empty.

Suppose $k > 0$. Apply a homeomorphism (in this problem we are only interested in crossing numbers and winding numbers so a homeomorphism that straightens out the chord will have no effect) so that the chord is a straight line from $(-1, 0)$ to $(1, 0)$ as in the diagram and apply a planar isotopy to the fork (note that the placement of the points is guaranteed by the definition of a chord). Denote the upper half disk by D_n^+ and the lower half disk by D_n^-:

(SEE n3diskdiagram.bmp)

By minimality of k, any arc of F that begins and ends on C lies entirely in D_n^-. This arc together with the segment of C must encircle p_3 (otherwise, we could reduce the number of intersection points with an isotopy that looks like the second Reidemeister move). Similarly, each arc of F which begins and ends on C must contain either p_1 or p_2 (or start at one of them) but not both since then we would have a closed loop around all the points.

Then if a_i and a_j are the winding numbers in the definition of $\langle h(F), C\rangle$, $a_j = a_i \pm 1$. Also, the crossing signs ϵ_i and ϵ_j are opposite signs:

(SEE windingnumbers.bmp)

Together, $e_j(-1)^{a_j} = e_i(-1)^{a_i}$. Since the set of all covering spaces is just the group generated by q which shifts the covering space up by one sheet, it is isomorphic to $\mathbb{Z}$ so using q^{-1} which maps to -1

$$\sum_{i=1}^{k} e_i(-1)^{a_i} = \pm k \neq 0$$

□

Finally,

Theorem. *The Burau representation of B_3 is not faithful.*

Proof. Let h be a homeomorphism in the kernel of the Burau representation. We will show that it is isotopic to the trivial homeomorphism.

Choose as a prong $P(F)$ a horizontal line segment from p_1 to p_2, a chord that does not intersect this prong, and the rest of the fork so that it does not intersect the chord. Then since $P(F)$ is disjoint from C, by Lemma 3, $\langle F, C\rangle = 0$. By Lemma 2, $\langle h(F), C\rangle = 0$ as well, and by Lemma 3 again, $P(h(F))$ is isotopic to a segment disjoint from C. By applying an isotopy, we can assume that $P(h(F)) = P(F)$.

If we do the same thing all over again, but on a permutation of p_1, p_2, p_3 then we can assume that h fixes all three of the segments that connect pairs of these three points. Applying an isotopy, h fixes the circle that contains all three points. But we know that h must be some kind of combination of rotations of puncture points into each other so this must mean it is some power of Dehn twist (i.e. it smoothly twists an annulus centered at the circle), as mentioned above. It can be shown that such a twist has representation $q^3 I$ where I is the identity matrix. The only possibility is that the representation of h is the 0th power, i.e. h is the non-twisting identity homeomorphism. □

Extending this to $n = 4$

It can be shown that Lemma 3 somewhat characterizes the n for which the Burau representation is unfaithful:

Proposition 4. *The following are equivalent:*

(1) *The Burau representation of* B_n *is faithful.*
(2) *The direct statement of Lemma 3 holds: if* F *and* C *are any fork and chord such that* $\langle F, C\rangle = 0$ *then* $P(F)$ *is isotopic to an arc disjoint to* C*.*

The author defines a standard form of the placement of F and C within D_n that is very similar to that used above. Ultimately, one can think of this standard form as encoding, not twisting operations directly, but a structure which exposes properties of homeomorphisms when acted upon by them.

Using this standard form, the algorithm's goal is to search for curves F and C in the disc which intersect, but which satisfy $\langle F, C\rangle = 0$ which is the violation to the above equivalence. Again, this method was applied in the $n = 5$ case to find explicit non-trivial braid in the kernel of the Burau representation of B_5.

BIBLIOGRAPHY

"Study of Braids", Kunio Murasugi and Bohdan I. Kurpita, 1999
"Mathematical Theory of Knots and Braids : An Introduction", Siegfried Moran, 1983
"Does the Jones polynomial detect the unknot?", Stephen Bigelow, 2002