A discussion of the Knot Quandle

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Def. A *quandle* is a set with two operations, called |> and |>$^{-1}$. These two operations satisfy the following three conditions.
Q1 x |> x = x
Q2 (x |> y) |>$^{-1}$ y = x
Q3 (x |> y) |> z = (x |> z) |> (y |> z)

Ex. One can associate a quandle to any group in the following way. The underlying set of the quandle consists of the elements of the group, and the operations |> and |>$^{-1}$ are defined using conjugation in a group:
|> : (x,y)---> $y^{-1}xy$ |>$^{-1}$: (x,y)--->yxy^{-1}

To show that this indeed defines a quandle, we must check that the two conjugation operations satisfy the three quandle conditions.
For Q1 we have:

x |> x = $x^{-1}xx = x$

Q2: (x |> y) |>$^{-1}$ y = ($y^{-1}xy$) |>$^{-1}$ y = $yy^{-1}xyy^{-1} = x$
Q3: (x |> y) |> z = ($y^{-1}xy$) |> z = $z^{-1}y^{-1}xyz$
(x |> z) |> (y |> z) = ($z^{-1}xz$) |> ($z^{-1}yz$) = $z^{-1}y^{-1}zz^{-1}xzz^{-1}yz$ = $z^{-1}y^{-1}xyz$

Thus to create a quandle from a group, you only need to take the underlying set of the group and the two operations of conjugation.

Also, you can generate a group from a quandle by taking the elements of the quandle modulo the relations of conjugation.

{ x , for x in Q | x |> y = $y^{-1}xy$ for x,y in Q}

Thus quandles and groups are closely related, giving a somewhat firmer basis from which to understand the quandle.

However, as we have defined so far the quandle is an abstraction, and to relate the quandle to a knot, we need to define explicit relations amongst the elements of the quandle.

The knot quandle is computed from a knot diagram in the following way. Label the arcs, always putting the labels on one side of the knot, then relate the elements as follows:

```
      b |
 a      |    c
---------|--------      a |> b = c
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```
      |  b
a     |       c
-------|-----------      a |>^-1 b = c
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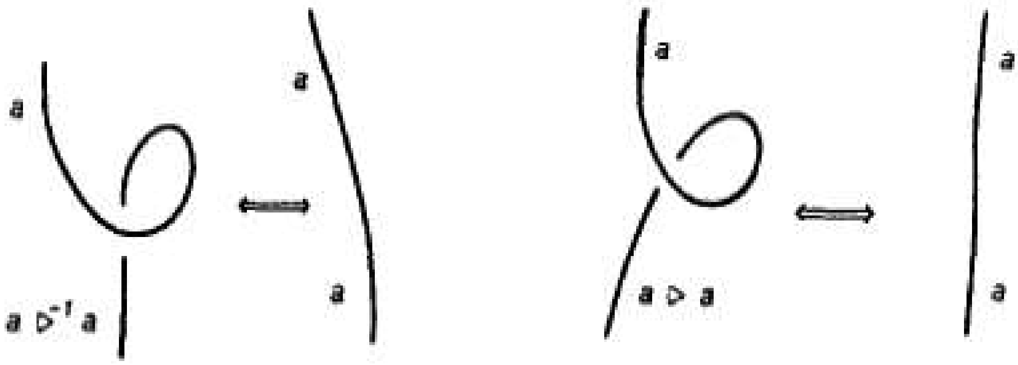
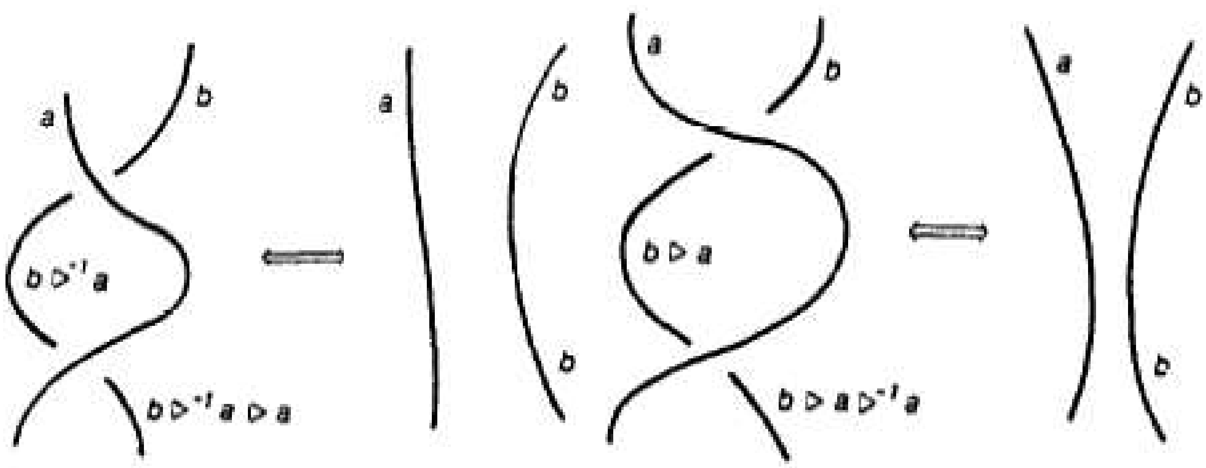
Now, to show that the knot quandle we have defined is an invariant of ambient isotopy, we must show that it is an invariant under Reidemeister moves.

For the first Reidemeister move we get:



$a \rhd^{-1} a = a$ $\qquad$ $a \rhd a = a$

For R2 we have:



$b \rhd^{-1} a \rhd a = b$ $\qquad$ $b \rhd a \rhd^{-1} a = b$

And for R3:

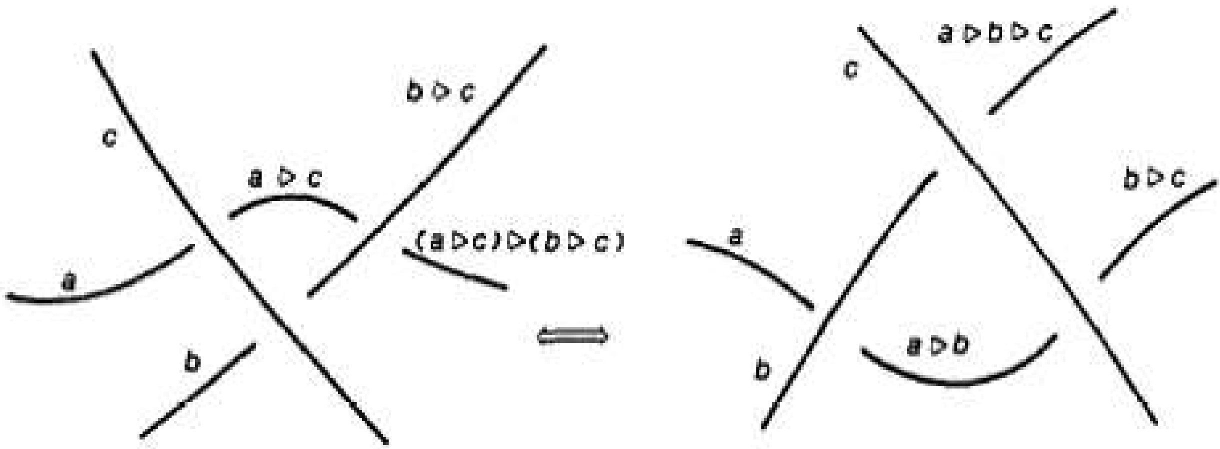
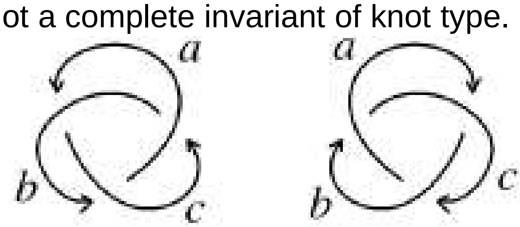


Fig. 9. $(a \triangleright c) \triangleright (b \triangleright c) = (a \triangleright b) \triangleright c.$

Thus it seems very convenient that all of the conditions initially set on the quandle make it an invariant under Reidemeister moves. It seems as though the main motivation in the relation of the quandle to knots, is to take something relatively unknown in the form of knots, and give them a certain degree of abstraction. The quandle is basically a labeling of the arcs and crossings, in such a way that it can be considered as an algebraic structure, and related to something well known (i.e., groups).
The quandle is not changed under Reidemeister moves, so for two equivalent knots, their quandles should be isomorphic. So if two knot quandles are isomorphic then the unoriented knots are equivalent. However, as the following example shows, the quandle is not a complete invariant of knot type.



$$Q = \langle a, b, c \mid a \sim b \triangleright c, b \sim c \triangleright a, c \sim a \triangleright b \rangle$$

Which shows two knots (the levt and the right trefoil) which are not equivalent, but which have isomorphic fundamental quandles.

From here we can compute some simple knot quandles. We have already done the trefoil, but we can do Whitehead link, or 4-1.

Knots that the quandle does allow us to distinguish are, for example, 5_1 and the unknot, and 6-3 and 5-1. We couldn't distinguish using three colorings.