The Mathematical Theory of Hitches

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Abstract:

The mathematical theory of hitches is the analyzing of the forces involved in the creation and holding of a hitch. By formulating mathematical representations of tension forces, turnings, pinches, and friction, the theory of hitches is able to express the fundamentals of how hitches work, and provide criteria for whether a hitch will hold or not and under what conditions.

I. Introduction

Hitches have been a popular tool to use fixed objects to oppose force for many years. Cowboys have hitched their horses to fences and sailors have hitched the sails to the ships that have created history. And while such importance may lie on one end of rope from a hitch, across the hitch the other end of rope may simply rest limp with no force whatsoever. Whether a hitch will allow such a disproportion of force depends on its topology. While three hundred years ago sailors may have known that an increase in turns or crosses within their hitch increases the strength, the breaking down of each component of a hitch into mathematical models has not been developed until recently. It is the purpose of this paper to explain how a hitch can be modeled mathematically, and how these models can be analyzed to determine whether a hitch will hold or not. Furthermore, it will be found that there are many other factors that play roles in the creation of a hitch's strength. The effects of these factors will be analyzed as well. First, however, it is necessary to understand the properties of a hitch.

II. Components of a Hitch

A *hitch* is composed by an assembly of different interactions between a rope and a pole. Wrapping a rope around a pole increases the amount of surface area, therefore, increasing friction. The friction force created by making turns around a pole is the first step in creating a disproportionate amount of forces between the ends of a hitch.

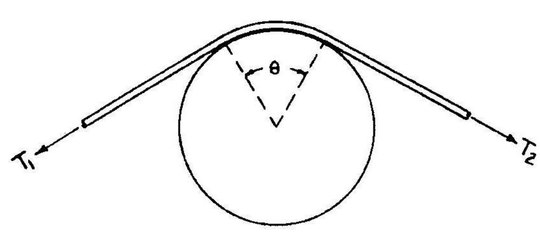

Figure 1: The friction of the rope on a pole creates a disproportion between tensions T_1 and T_2.

Therefore, the first step in analyzing hitches is representing the amount of friction created by the wrapping of rope around the pole. A segment of a hitch is defined as the piece of rope that begins beneath one over-crossing and continues until it travels beneath another over-crossing. The free segment, denoted T_0, begins at the loose end of the rope and ends when it passes under the hitch's first crossing. The next segment, denoted T_1, then travels until passing beneath the next pinch. The new segment then begins on the other side of the over-crossing. Segments are created in this manner, beneath a crossing, until the last segment exits the hitch and connects the load.

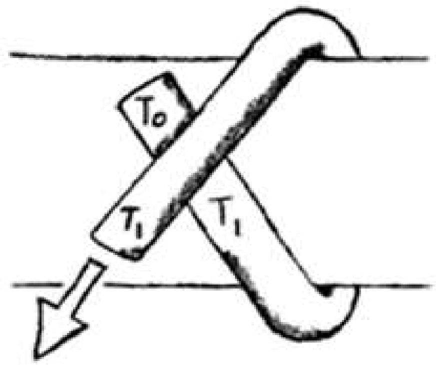

Figure 2: Segments are created by the rope passing beneath itself, pinching the rope.

The static frictional force is the force that is exerted by the friction between the rope and pole. Static friction is obtained only when the rope does not move and the net force is zero. There is a coefficient of static friction, denoted μ, between the rope and the pole that marks the extent of the friction between the two surfaces. When the coefficient of static friction is multiplied by the normal force, the result is the maximum force that can be applied to the system without the system moving. If the rope is moving, the static friction becomes kinetic friction in the system. The coefficient of static friction will change depending on the type of rope and pole. As the rope produces a force onto the pole, the normal force, denoted N, is exerted against the rope. Thus, our first condition for whether a rope will slip or whether the hitch will hold becomes

$$T_2 \leq T_1 \mu N, \qquad (1)$$

Where T_1 is the tension in segment one, and T_2 is the tension in segment two. Here, if T_2 is larger than T_1 then the rope will not move if the tension in T_2 is equal to the product on the right side of the inequality, and will move in the direction of T_1 if T_2 is found to be smaller than the right side.

This inequality, however, does not incorporate the varying degrees of friction due to surface area. Therefore, it is more accurate to represent the inequality as below, where θ represents the angle corresponding to the arc length where the rope is in contact with the surface area (Figure 1).

$$T_2 \leq T_1 e^{(\mu\theta)}. \qquad (T_2 > T_1) \qquad (2)$$

Because θ is a multiple of 2π, the inequality is simplified if written as

$$T_2 \leq T_1\varepsilon^n \qquad \text{(n = number of turns).} \qquad (3)$$

Here

$$\varepsilon = e^{(2\pi\mu)}.$$

Therefore, the larger number of turns increases the friction and allows for the hitch to still hold as T_2 becomes larger. We note that we consider the weight of the rope negligible and exclude the normal force in our model, thus, we will only take into account the tension of the force. Inequality (1) represents only the properties of friction and turns within a hitch. Therefore, other equations must be included when describing hitches that include more than just turns.

Often, after a turn, the rope will cross itself. Now, not only friction between the rope and the pole influences the strength of the hitch, but the force of one rope on the other has created a pinching effect that adds considerable strength to the hitch.

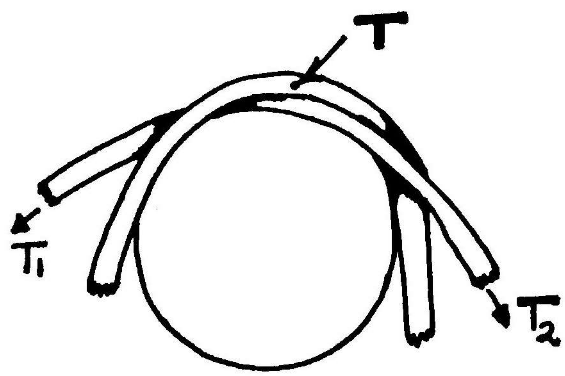

Figure 3: The tension T pinches the rope below it creating a disproportion of tensions and dividing segments.

As the rope passing over the lower rope, T, presses down onto the lower rope, a force is created at this point that divides the lower rope into two sections. The force exerted by T creates an inequality between the two divided parts. The extent of this inequality is proportional to the tension in T. The corresponding inequality can be written as

$$T_2 \leq T_1 + \eta T. \qquad (T_2 > T_1) \qquad (4)$$

Here, η depends on the coefficient of friction between the two touching ropes and their diameters. Further discussion of the effects of the ratio of diameters is included later.

It is also important to note that an assumption has been made that the tension in T does not change when it crosses. This assumption is inaccurate, but is made for simplicity. The effects of crossings are likely to be small in relation to the effects due to the tensions. Now, however, the effects of crossings, as well as turns and friction can be solved for by the known inequalities we have derived. Let us view a classic example of a hitch including the properties we are currently aware of.

III. Analyzing a Hitch

Due to the turning and crossings of a hitch, as we continue along the rope from one end to the other we can create a chain of inequalities relating the tension in different sections of a hitch. Following an example from Bayman's article [1], we will analyze the clove hitch.

<u>Example:</u> Consider the clove hitch (see figure 4 below).

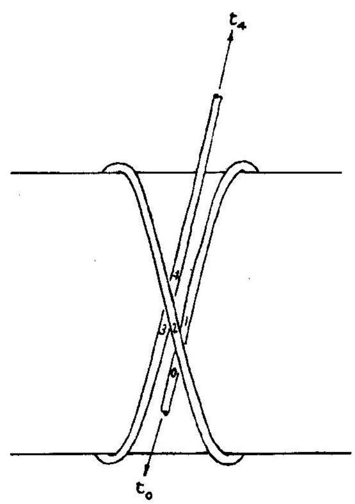

Figure 4: The Clove Hitch

From Figure 3 it can be seen that the following inequalities are true

$$T_0 \leq T_1 \leq T_2 \leq T_3 \leq T_4.$$

Analyzing the clove hitch using the above conditions for turning and crossings, we obtain a system of inequalities that must be satisfied for the hitch to hold that resemble

$$T_1 \leq T_0 + \eta T_2, \qquad (5a)$$

$$T_2 \leq \varepsilon\, T_1, \qquad (5b)$$

$$T_3 \leq \varepsilon\, T_2, \qquad (5c)$$

$$T_4 \leq T_3 + \eta T_2. \qquad (5d)$$

Combining the equations above, we obtain

$$T_2(1-\eta\ \varepsilon) \leq \varepsilon\ T_{0.} \qquad (6)$$

The value in the parentheses on the left hand side can be either negative or positive depending on whether the value $\eta\varepsilon$ is greater than or less than 1. The inequality

$$\eta\varepsilon < 1, \qquad (7)$$

means that the friction and its results are very small. This condition causes the left hand side of inequality (6) to be positive. The answer to whether the hitch will hold or not now rests solely on the relation between T_4 and T_0. If we combine the above equations for this system, we see that the relation between T4 and T0 rests in the equation

$$T_4 \leq T_0\,[\varepsilon(\varepsilon + \eta)/(1 - \varepsilon\eta)]. \qquad (8)$$

Therefore, the hitch will hold as long as T_4 and T_0 are related as in (8).

On the other hand, if $\eta\varepsilon > 1$, then the strong presence of friction will affect the conditions of the hitch. The value in the parentheses in equation (6) will be negative, and therefore the entire quantity on the left side will be negative. Thus, for all values of T_2, the inequality will hold, as will the hitch. This is true even if no tension is on T_0 and for large amounts of tension on T_2.

IV. Matrix Analysis of a Hitch

The above conditions can also be derived using a matrix representation of the inequalities. An analysis of the Ground Line Hitch where we place the derived inequalities into matrix form is now presented. Later, a general approach to using matrices to analyze various hitches is described.

<u>Example:</u> We will consider the Ground Line Hitch.

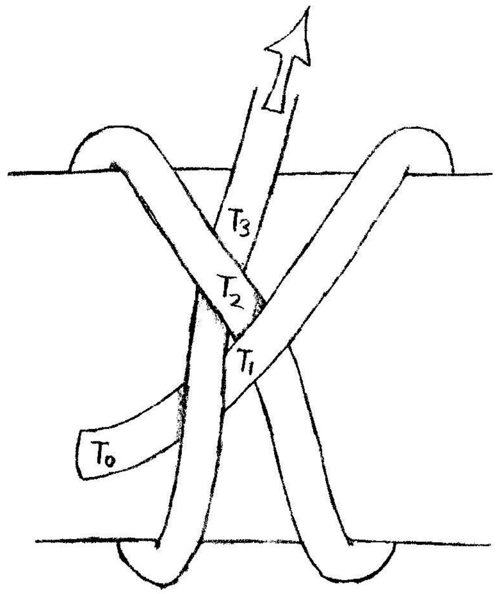

Figure 5: The Ground Line Hitch with tensions of respected segments.

The inequalities obtained by the method above are

$$T_1 \leq T_0 + \eta\varepsilon T_{2,} \qquad (9a)$$

$$T_2 \leq \varepsilon T_1 + \eta T_{1.} \qquad (9b)$$

Here, only the tensions for the first and second segments are of concern because the initial segment, T_0, and the final segment, denoted T_3, are omitted since they bear the end loads. Since T_0 is omitted, inequality (9a) can be written as

$$T_1 \leq \eta\varepsilon T_2. \qquad (9c)$$

Placing each inequality less than or equal to zero we obtain

$$T_1 - \eta\varepsilon T_2 \leq 0, \qquad (10a)$$

$$T_2 - \varepsilon T_1 - \eta T_1 \leq 0. \qquad (10b)$$

Taking the coefficients for each inequality with respect to T_1 and T_2 we find the coefficients

T_1	T_2
1	$-\eta\varepsilon$
$-\varepsilon - \eta$	1

In matrix form, we obtain

$$\begin{bmatrix} 1 & -\eta\varepsilon \\ -\varepsilon-\eta & 1 \end{bmatrix}$$

In the prior examples not using matrices, the inequalities are solved plugging one into another and finding the necessary conditions for the coefficients in order for the system of equations to be solved. Using coefficient matrices, we are able to find the determinant. So long as the determinant of the matrix does not equal zero, the linear system of equations is solvable for more than the trivial solution. Thus, once we have found the determinant, we can set it equal to zero and find for what values of η the system will be solvable for.

Solving for the determinant of this matrix we find it to be

$$1 - \eta\varepsilon(\eta + \varepsilon). \qquad (11)$$

Therefore, the matrix is solvable for more than the trivial solution and the hitch will hold so long as

$$\eta\varepsilon(\eta + \varepsilon) \neq 1. \qquad (12)$$

It is noted that the hitch will also hold if $\eta\varepsilon(\eta + \varepsilon) = 1$, however, this implies there must be no tension in any of the segments of the hitch. Thus, we concern ourselves with the condition $\eta\varepsilon(\eta + \varepsilon) \neq 1$ since this is more concerned with real life situations where one end of the hitch has a tension and the other does not. We see that so long as η and ε abide by the inequality $\eta\varepsilon(\eta + \varepsilon) > 1$, the determinant will be less than zero and the hitch will hold no matter what tension is T_0. If the determinant is greater than zero ($\eta\varepsilon(\eta + \varepsilon) < 1$), the system may still be solved, however, the solution is dependent on the tension in T_0. We will look more into these conditions during the general matrix section. Here we will concern ourselves with solving for $T_0 = 0$. Thus the hitch will hold so long as $\eta\varepsilon(\eta + \varepsilon) > 1$.

Additionally, because ε (which is equal to $e^{(2\pi\mu)}$) is always greater than 1, we are able to find a range for η where

$$[\varepsilon(\eta + \varepsilon)]^{-1} < \eta < \varepsilon^{-1}. \qquad (13)$$

While η is within this range, the hitch will hold.

V. The General Case for a Hitch

Finding a system of inequalities for a hitch and analyzing its matrix representation enables easy analysis to determine whether a hitch will hold. Furthermore, it allows for an easy way to determine under what conditions, with respect to the friction coefficients, a hitch will hold. It is therefore beneficial to generalize the matrix form to be used for all hitches. Above, we witnessed an example for the Ground Line Hitch. The example assumes that the tension in T_0, the free end, was zero. This is why the inequalities within the matrix were set to be less than or equal to zero. Following a generalized matrix form presented by Bayman [1], however, it allows the evaluation of a hitch with different tensions resulting at the T_0 end. To analyze a hitch in this way it is necessary to divide the hitch into segments as is done above. We will again refer to the free end as T_0 and label every segment thereafter with an increasing index of T. A new segment begins beneath every crossing and T_i will represent the tension in the beginning of segment i.

Recall from above that ε^n is a friction force due to the number of turns, n. Here ε^{ni} denotes the friction as before, where

(a) n_i is the number of turns in the i^{th} segment.

It follows that more turns will create more friction and, therefore, allow the segment to bear more tension. In general, $\varepsilon^{ni}(T_i)$ is the tension at the end of segment i. Let

(b) b_i be the number of the segment under which the i^{th} segment begins, and allow

(c) m_i to be the number of turns from the beginning of the b_i segment to where it crosses over the i^{th} segment.

Therefore, the m_i variable is the same as the n_i variable only for the b_i^{th} segment rather than the i^{th} segment. It follows that $\varepsilon^{mi}(T_{bi})$ is the tension of the b_i^{th} segment after the m_i^{th} turn.

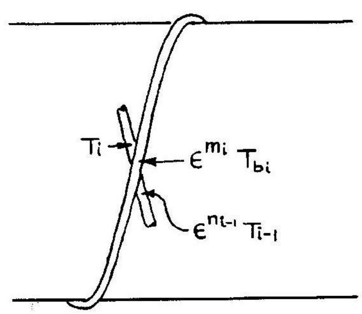

Figure 6: The general case for segments.

To find the tension in any segment we can generalize an inequality for T_i as

$$T_i \le \varepsilon^{ni-1}T_{i-1} + \eta\varepsilon^{mi}T_{bi}. \qquad (T_{i-1} \le T_i) \qquad (14)$$

Here, i = 1, 2, 3, . . . , q. Where q represents the number of the last segment found in the hitch.

Rather than find all of the equations and plug one into another until a ratio is found correlating T_0 and T_q, we can place the coefficients of the inequalities into matrix form to find matrix A. We can then rewrite (14) as

$$\sum_{j=1}^{q} A_{ij} T_j \leq \delta_{i,1} T_0 \qquad (T_{i-1} \leq T_i), \tag{15}$$

Where A is the matrix written as

$$A_{ij} = B_{ij} - \eta C_{ij}, \tag{16a}$$

Here, B and C are defined as

$$B_{ij} = \delta_{ij} - \varepsilon^{nj}\delta_{i-1,j,} \tag{16b}$$

$$C_{ij} = \varepsilon^{mi}\delta_{bi,j}. \tag{16c}$$

The Kronecker delta, δ_{ij}, is defined as $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

After finding the determinant of matrix A, we can set it equal to zero and solve for what conditions must be met for the hitch to hold. As long as the determinant is greater than zero, the hitch will only hold under correct conditions for η, and it the hitch will not hold if there is no tension in the T_0 segment. If the determinant of matrix A is less than zero, the hitch will then hold even if there is no tension in the T_0 segment. The η that causes the determinant of matrix A to equal zero is denoted as our critical coefficient of friction, η_c. Setting $\eta = 0$, it is evident that A = B and det A = 1. The above properties are shown as

$$\det A > 0 \qquad \text{if } 0 \leq \eta < \eta_c, \tag{17a}$$

$$\det A = 0 \qquad \text{if } \eta = \eta_c, \tag{17b}$$

$$\det A < 0 \qquad \text{if } 0 \leq \eta_c < \eta. \tag{17c}$$

For $0 \leq \eta < \eta_c$: We will now show that for the case of (17a), conditions must be imposed for η in able for the hitch to hold.

Within the range of η in (17a) we denote M to be the reciprocal of matrix A. Therefore, it can be seen that

$$MA = M(B - \eta C) = (B - \eta C)M = 1 \tag{18}$$

Though it can be seen in (16b) and (16c) that B and C do not rely on η, M does rely on η. We can find the entries of M if we set $\eta = 0$. Therefore for the row i, and column j, we can find that the entries are as follows

$$M_{ij}(0) = 0 \qquad \text{if } i < j, \tag{19a}$$

$$M_{ij} = 1 \qquad \text{if } i = j, \tag{19b}$$

$$M_{ij} = \varepsilon^{nj + n(j+1) + \ldots + n(i-1)} \qquad \text{if } i > j. \tag{19c}$$

Therefore, all matrix entries of $M_{ij}(0)$ are non-negative. From (16b) and (16c) it can also be seen that B and C are non-negative. Thus, differentiating (18) with respect to η, we find

$$dM/d\eta\ (A) = MC,$$

$$dM/d\eta = MCM. \qquad (20)$$

Thus, $dM(0)/d\eta$ is also non-negative. This implies that M is increasing and if η increases from zero, no M entry will decrease below their $\eta = 0$ values found in (19). Therefore so long as $0 \leq \eta < \eta_c$, M exists and is continuous. Thus,

$$M_{ij}(\eta) \geq 0 \qquad \text{if } i > j \qquad (21a)$$

$$M_{ij}(\eta) \geq 1 \qquad \text{if } i = j \qquad (21b)$$

$$M_{ij}(\eta) \geq \varepsilon^{nj + n(j+1) + \ldots + n(i-1)} \qquad \text{if } i < j \qquad (21c)$$

Also, as a result to (19), we find that as the entries of the matrix M move from left to the right, they increase. Furthermore, as the entries move from the bottom to the top they also increase. Therefore,

$$M_{i+1,j}(0) - M_{ij}(0) \geq 0, \qquad (20a)$$

$$M_{i,j-1}(0) - M_{ij}(0) \geq 0. \qquad (20b)$$

Again, since no $M_{ij}(0)$ entry can be non-negative, so long as $0 \leq \eta < \eta_c$ the inequalities from (22) can also be written as

$$M_{i+1,j}(0) \geq M_{ij}(0), \qquad (23a)$$

$$M_{i,j-1}(0) \geq M_{ij}(0). \qquad (23b)$$

These inequalities show that there is no entry in M that is larger than the entry in the top right corner, M_{q1}.

The Riemann sum for the matrix M is

$$\sum_{j=1}^{q} A_{ij}$$

Multiplying (15) by the Riemann sum for matrix M we find

$$\sum_{k=1}^{q} M_{ki} \sum_{j=1}^{q} A_{ij} T_j \leq T_0 \sum_{k=1}^{q} M_{ki} \delta_{i,1} = T_0 M_{k1} \qquad (24)$$

Because M is the reciprocal of A, the two Riemann sums on the left hand side of (24) cancel and we obtain the inequality

$$T_k \leq T_0 M_{k1}. \qquad (0 \leq \eta < \eta_c) \qquad (25)$$

As shown in (21) and (23a), M_{k1} is not negative. This implies that

$$T_k > 0, \qquad (26a)$$

$$T_{k+1} > T_k. \qquad (26b)$$

Therefore, it is found that (25) is an inequality that is necessary for the hitch to hold. This is the general representation for what we have already found for the clove hitch in (15). This inequality is necessary for the hitch to hold in all low friction cases such as the low friction case (7) for the clove hitch. Because a multiple of the free end, T_0, is required for the hitch to hold when $0 \leq \eta < \eta_c$, if $T_0 = 0$, then the only T_k that satisfies the inequalities (25) and (26a) is when $T_k = 0$. Thus, if there is no tension on the free end, the only way for the hitch to hold is when there is also no tension on the last segment for all η, such that, $0 \leq \eta < \eta_c$. Thus, there is no tension at all upon the hitch. However, when $\eta \geq \eta_c$, the hitch will hold if there is a tension in the last segment even when there is no tension on the free end.

For $0 \leq \eta_c < \eta$: We will now show that tension is not necessary in the T_0 segment for the hitch to hold under the conditions of (17c).

Recall that the cofactor of a matrix element A_{ij} is defined as $(-1)^{i+j}\det(\tilde{A}_{ij})$, where $\tilde{A}$ is the matrix that results when canceling the row and column from matrix A in which the element A_{ij} is found. Let $\eta \geq \eta_c$, and $a_{ij}(\eta)$ be the cofactor of $A_{ij}(\eta)$, so that

$$\sum_{j=1}^{q} A_{ij} a_{kj} = \sum_{j=1}^{q} A_{ij} (\tilde{a})_{jk} = \delta_{ik} \det A. \qquad (27)$$

By dividing (27) by det A we find

$$M_{kj}(\eta) = a_{jk}(\eta)/\det A. \qquad (28)$$

If we enforce the conditions that $0 \leq \eta < \eta_c$, the determinant of A is greater than zero as stated in (17a). Furthermore, as before, the conditions of M would imply that the cofactor entries also increase or are equal as the position of the entry moves to the right and up. Therefore, similarly to M, no entry, $a_{jk}(\eta)$, is greater than the upper right entry $a_{1q}(\eta)$. We can write these inequalities as

$$a_{jk}(\eta) \geq 0 \qquad (29a)$$

$$a_{j,k+1}(\eta) \geq a_{jk}(\eta) \qquad (29b)$$

$$a_{j-1,k}(\eta) \geq a_{jk}(\eta) \qquad (29c)$$

when $0 \leq \eta < \eta_c$.

Under the condition where $\eta = \eta_c$, det A = 0. We can then define a set T_j as

$$T_j = a_{1j}(\eta_c). \quad (30)$$

It is now possible to rewrite (27) as

$$\sum_{j=1}^{q} A_{ij}(\eta_c)T_j = 0, \quad (31)$$

If $T_0 = 0$, it is evident that equation (31) is the same as equation (15). So long as there is no tension in the free end, and $a_{1q}(\eta) \neq 0$ which implies that not all $a_{ij}(\eta) = 0$, then T_j defined in (30) are tensions that satisfy the conditions for the hitch to hold if and only if $\eta \geq \eta_c$.

In the condition that all entries $a_{ij}(\eta_c)$ vanish, the determinant of A and all the cofactor entries are polynomials of η. Therefore, it can also be shown that the hitch will hold as all $a_{ij}(\eta_c)$ vanish.

To represent $a_{1q}(\eta)$ as a polynomial, we can write

$$a_{1q}(\eta) = (\eta - \eta_c)^r\alpha(\eta), \quad (32a)$$

where

$$\alpha(\eta_c) \neq 0, \qquad r \geq 1. \quad (32b)$$

As stated above, when $0 \leq \eta < \eta_c$, no other $a_{ij}(\eta)$ will exceed $a_{1q}(\eta)$. Therefore, all $a_{ij}(\eta)$ can be written in the form of (32). And the resulting power of $(\eta - \eta_c)$ will be at least as large as r. Furthermore, because there are a q number of segments, the dimension of *a*, is q. Thus, if we find the determinant of *a* as a polynomial of η where η_c is a zero, we find

$$\det a = (\eta - \eta_c)^s\gamma, \quad (33a)$$

where

$$s \geq qr. \quad (33b)$$

Meanwhile, taking the determinant of (27) it can be seen that

$$\det a = (\det A)^{q-1}. \quad (34)$$

Comparing (33) and (34) it is seen that with a power of $(\eta - \eta_c)$ greater than r the det A will vanish when $\eta = \eta_c$. Thus, taking (23) and defining now

$$T_j(\eta) = a_{ij}(\eta)/(\eta - \eta_c)^r, \quad (35)$$

we can show taking limits that as $\eta \rightarrow \eta_c$, the following is true; (i) $T_q(\eta)$ approaches a finite positive value $\alpha(\eta)$, (ii) $T_{i(<q)}(\eta)$ approaches either a positive value or zero, and (iii) det $A/(\eta - \eta_c)^r$ approaches zero.

Dividing (27) by $(\eta - \eta_c)^r$, we find

$$\sum_{j=1}^{q} A_{ij}(\eta_c)\, T_j(\eta_c) = 0 \qquad [T_j(\eta_c) \leq T_{j+1}(\eta_c)]. \tag{36}$$

Thus, it can now be seen that for all $\eta \geq \eta_c$, there are tensions T_j, such that the conditions are satisfied and the hitch will hold when $T_0 = 0$.

Therefore, under the condition that $0 \leq \eta < \eta_c$, we showed that an a correct ratio of tension was necessary between the initial free segment and all other segments, including segment q, in order for the hitch to hold. However, if $\eta \geq \eta_c$, then the tension on the free end of the hitch is arbitrary, even unnecessary.

Due to the nature of the matrix there are many abbreviations that can be made to simplify the computations. Because the last segment in the hitch is segment q, we know that there will be q amounts of columns and rows within the matrix representing the hitch. Therefore, the matrix A has dimension q. However, no segment ever begins under the w segment. Thus, the q column in matrix A has a value one in the diagonal component, and q-1 zeros over the rest of the column. Since this implication has no affect on the determinant, the last column, column q, can be deleted from the matrix as well as the corresponding row. We maintain the square matrix and its computational advantages, while deleing any unnecessary computations. This result can also be placed on any segment, i, not just q, that does not cross over any other segment. This concern of only segments that pass over other segments leads to another, equivalent, condition to find η_c,

$$\det E(\eta_c) = 0, \tag{37a}$$

where

$$E_{ij}(\eta) \equiv \delta_{ij} - \eta \sum_{k=1}^{i} \epsilon^{m_k+n_k+n_{k+1}+\cdots+n_{i-1}}\, \delta_{b_k,j}. \tag{37b}$$

Here, E is the matrix which represents only segments that cross over one another. Equation (37) is beneficial because often the dimension of A is much larger than the dimension of E [1].

VI. Example Computations of Hitches

Having obtained the calculations for a general hitch, it is now much easier to calculate the critical numbers of many hitches. Although many hitches are very similar to the clove hitch derived above, there are also many hitches that have slight deviations, and even more hitches that are significantly more complicated. Below are examples of two hitches that have slight deviations from the clove hitch, but are still able to be computed using the general equation for hitches.

Here, we will use the above general matrix form to compute conditions for the Rolling Hitch:

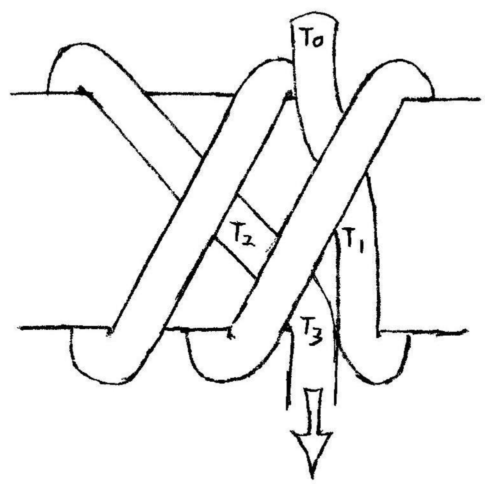

Figure 7: A Rolling Hitch with tensions for respected segments.

To find (15) we must first find (16a). For the first entry for matrix A, we set i = 1 and j = 1 for equation (16a). Therefore,

$$A_{11} = B_{11} - \eta C_{11}. \qquad (38a)$$

$$B_{11} = \delta_{11} - \varepsilon^3\delta_{0,1,} \qquad (38b)$$

$$C_{11} = \varepsilon^1\delta_{1,1}. \qquad (38c)$$

Thus,

$$B_{11} = 1, \qquad (39a)$$

$$C_{11} = \varepsilon, \qquad (39b)$$

Therefore, the first entry of matrix A becomes

$$A_{11} = 1 - \eta\varepsilon. \qquad (40)$$

In similar computations the rest of the entries for the matrix can be found. For matrix A we find

$$\begin{bmatrix} 1-\eta\varepsilon & 0 \\ -\varepsilon-\eta\varepsilon^2 & 1 \end{bmatrix}$$

Finding det A = $1 - \eta\varepsilon$, we notice that when $\eta\varepsilon \geq 1$, the hitch will hold. Furthermore, we find that the critical friction force is satisfied when $\eta \geq 1/\varepsilon$.

As another example we will analyze the Ossel Knot.

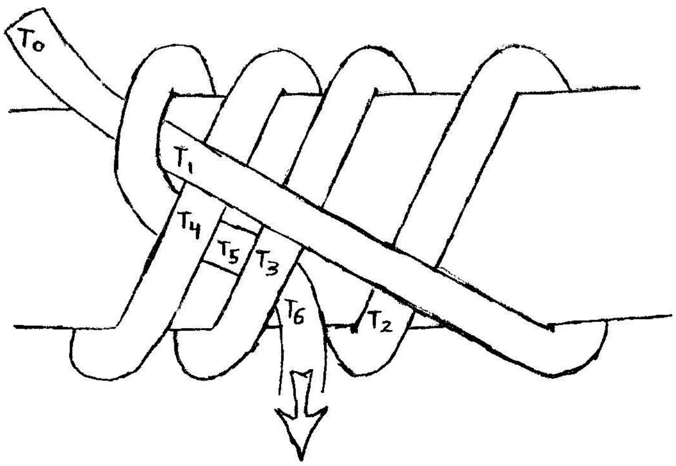

Figure 8: An Ossel Knot with tensions for respected segments.

Again for the first segment, i = 1, and for the first entry in row one of matrix A, j = 1. Thus, we solve again for equation (16) and find

$$B_{11} = 1, \qquad (41a)$$

$$C_{11} = 0. \qquad (41b)$$

Thus, the first entry in the matrix is

$$A_{11} = 1. \qquad (42)$$

Deriving the rest of the matrix entries in the same manner, we find matrix A to be

$$\begin{bmatrix} 1 & 0 & 0 & -\eta\varepsilon & 0 \\ -\varepsilon - \eta\varepsilon & 1 & 0 & 0 & 0 \\ -\eta\varepsilon & -\varepsilon & 1 & 0 & 0 \\ -\eta\varepsilon & 0 & -\varepsilon & 1 & 0 \\ 0 & 0 & 0 & -\varepsilon - \eta\varepsilon & 1 \end{bmatrix}$$

Solving for the determinant we find

$$\det A = 1 - \eta^2[\varepsilon^3 - \varepsilon + 1]. \qquad (43)$$

Thus, if $\eta^2[\varepsilon^3 - \varepsilon + 1] \geq 1$ the hitch will hold. Furthermore, the critical friction force is satisfied when $\eta \geq [(\varepsilon^3 - \varepsilon + 1)^{-1}]^{1/2}$.

VII: Application to Knots: A Simple Computation

Substituting the pole from hitches with rope and treating this system similar to a hitch allows for a mathematical representation of knots, or hitches upon themselves. Often these knots are used to connect two or more pieces of rope together and, therefore, raise the same question as before: what is the condition under which the knot will hold.

Many of the same principles which apply with hitches also apply to knots. As in hitches, the tension in each segment of the knot must decrease between segments in order for the knot to hold. This creates an inequality between the load bearing tension, T_q, and the free end which has no tension. However, whereas a hitch produces these inequalities by crossing itself and turning around a solid non-bending pole, knots cross and turn around other segments of rope that also cross and turn. This creates the situation in which a segment may be pinched or turned by many parts of rope, not just one pole.

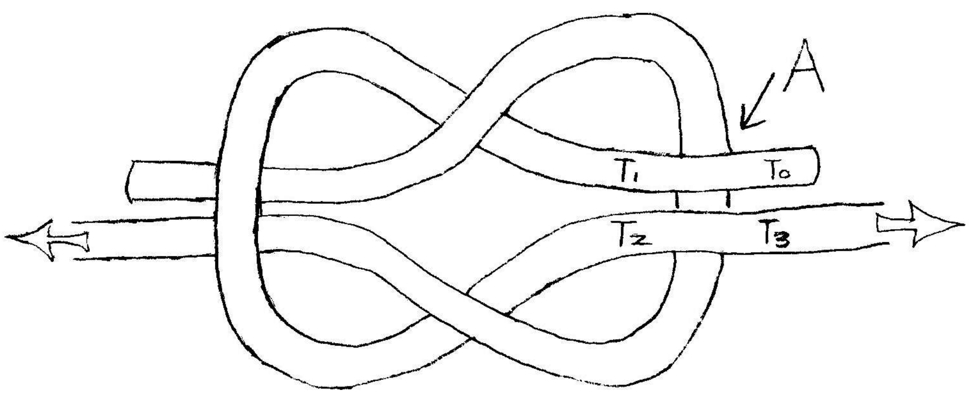

Figure 9: A Square Knot.

Following the work of Maddocks and Keller [5] and analyzing the symmetric Square Knot (Figure 9), it can be seen that at point A, when the knot is tightened, segment one will make a half turn around itself and segment two. Therefore, we find that

$$T_1 \leq T_0 + 2\mu T_1 e^{(\pi\mu)}. \qquad (44)$$

Here we use $e^{(\pi\mu)}$ rather than $\varepsilon = e^{(2\pi\mu)}$ because segment one makes only half a turn around itself. The factor of two is included because the tension in segment one is affected by two other segments of rope at point A. Furthermore, because we want conditions such that the free end bears no weight, $T_0 = 0$ we can rewrite (44) as

$$T_1 \leq 2\mu T_1 e^{(\pi\mu)}. \qquad (45)$$

Dividing both sides of (45) by T_1 and solving for μ, we obtain the rope coefficient of friction requirements for the knot to hold as

$$\mu \geq 1/2e^{(\pi\mu)}. \qquad (46)$$

Thus, the basic principles of hitch theory can be applied to more than just a rope around a pole. Many knots that have similar qualities to hitches can also be analyzed in this manner [5].

VIII: Other Factors Involved in Hitches

An understanding of the fundamentals of mathematical hitch theory has now been achieved. However, the mathematical representation of hitches continues to grow and become more refined and complex. Though the fundamental principles remain today, many variables within the hitch are still under analysis. We shall now explore some of the variables that can presently be understood and are being researched.

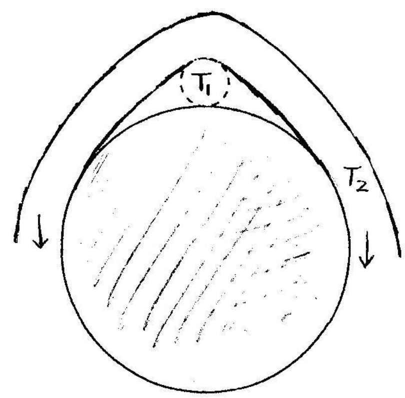

Figure 10: Side view of a rope crossing another rope.

As shown above, the process of one segment passing over another will cause one segment of rope to be slightly lifted from the pole. Therefore, the contact between the rope's segment and the pole is decreased. The extent of the decrease of contact with the pole can vary from simply a decrease in the normal force while the rope still touches, to lifting the rope completely from the pole. Whatever the degree, the decrease in rope contact will decrease the friction. Therefore, whether the hitch will hold or not is dependent on what degree the rope's contact with the pole is decreased due to a crossing. The extent of this effect is dependent on the diameters of the pole and rope, as well as the tensions. Within the model above, the effects due to rope crossings are minimized by stipulating that the rope's diameter must be much less than the diameter of the pole.

Another factor not included in our model above is the weight of the rope. If the rope is very heavy, its own weight will influence what force is necessary to cause the hitch to slip. In our model, we neglected the weight of the rope so as to neglect the normal force from our calculations. The tension in a segment within our model, therefore, depends only on coefficients of friction, angle of rope to pole contact, and crossings. If the weight of rope is included, calculations must be made to predict the tension necessary to pull the rope due to its weight along the pole, and due to the weight of segments crossing over other segments.

Though the model above describes the fundamental elements within hitches such as crossings and turns, there are many hitches that consist of more complicated twists and turns. The Figure Eight Hitch (figure 11), for example, consists of a twist around the rope itself as well as a segment of rope passing between the pole and two other segments of rope. The effects of this type of property are unknown.

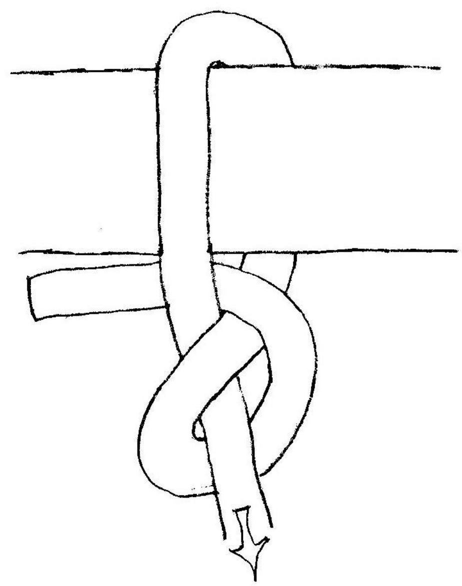

Figure 11: A Figure Eight Hitch.

The Snuggle Hitch (figure 12) also contains an interesting property in which it has a crossing atop another crossing. These multiple crossings not only introduce another element of friction, but also add to the tension force on the bottom strand, and therefore, increase the disproportion of tensions on the bottom segments. Furthermore, as noted above, the increase in angle for the top segment to pass over more than one diameter of rope can affect the amount of contact between the rope and the pole, thus, decreasing friction.

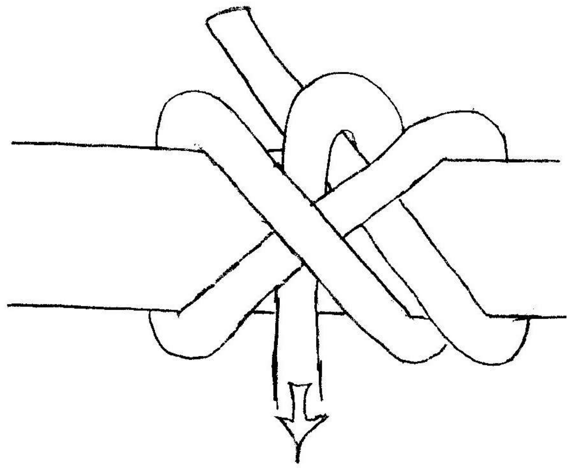

Figure 12: A Snuggle Hitch.

The Münter Hitch (figure 13) is often used by rock climbers to belay other climbers. The free end of the hitch can be moved, thus, allowing an increase or decrease in friction due to its orientation. In the model above, we concerned ourselves only with full turns. Analyzing the effects from fractions of turns would allow for varying representations of hitches dependent on the exact orientation of the free and loaded ends, which is omitted in our model.

Figure 13: Left: A Münter Hitch with low friction. Right: A Münter Hitch with high friction.

Other problems include hitches formed around objects that are not cylinders, two or more segments passing beneath one crossing, thus, sharing the tension of the crossing segment (this is seen in the clove hitch, although not taken into considerations during computations), and many other varying factors found in hitches.

IX: Conclusion

The principals behind what cause a hitch to hold or not have been mathematically analyzed and a model has been created. While the present theory on whether a hitch will hold focuses on the dominating aspects of the hitch such as the friction created by turns and crossings, and thus has helped us to understand the dominating factors within a hitch, there are many other factors that continue to be studied and incorporated into a model to represent the nature of hitches. Thus, the theory of hitches continues to expand and become more accurate as different principals of hitches are understood with mathematics.

References

[1] Bayman, B. F. *Theory of Hitches*, Amer. J Phys., 45 (1977), pp. 185-190
[2] Budworth, G. *Knots and Ropework*, Annes Publishing, (2003).
[3] Graydon, and H Hanson, E.D. *Mountaineering: Freedom of the Hills*, The Mountaineers, (1997).

[4] Kauffman, L. *Knots and Physics*. Teaneck, NJ: World Scientific, pp. 323-331, 1991.
[5] Maddocks, J. H, and J. B. Keller. *Ropes in Equilibrium*, SIAM J Appl. Math., 47 (1987), pp. 1185-1200