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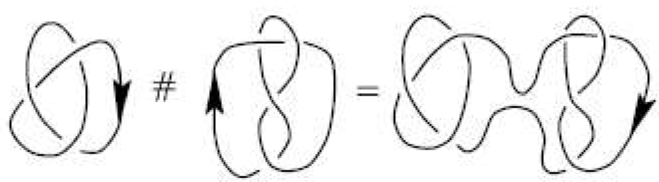
Prime Factorization of Knots

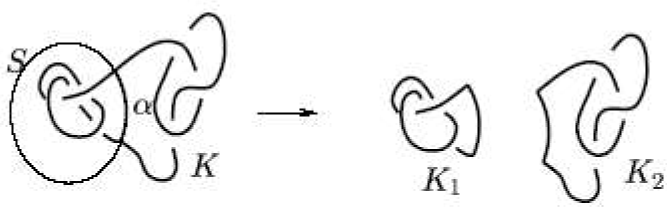
Introduction

The idea of knot factorization arose quite naturally while Alexander and Briggs were trying to classify all knots (see *On types of knotted curves*, Ann. Math., **28** (1926/27), 562-586). They noticed that one does not need to classify all knots, but rather only those knots that cannot be made up of smaller pieces. This motivates the following definitions:

Definitions: <u>Connected Sum</u>: *To form the connected sum of two knots, cut each knot at any point and join the boundaries of the cut, keeping orientations consistent. Notice that the connected sum is independent of the location of the cut.*



<u>Factoring</u>: *To factor a knot into two components, select a sphere that intersects the knot (transversely) at two points, and separate into two components. Then join the two loose ends of each knot with some path in the sphere. Notice that unlike the connected sum, this operation depends on the sphere that one chooses to factor the knot.*



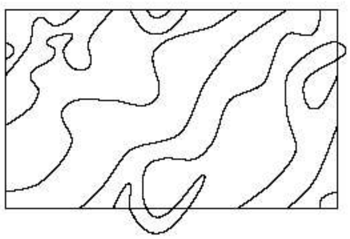
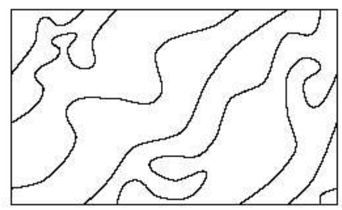
A knot is called <u>prime</u> if it can not be represented as a connected sum of two knots such that both of these are knotted. Any knot which is not prime is called <u>composite</u>.

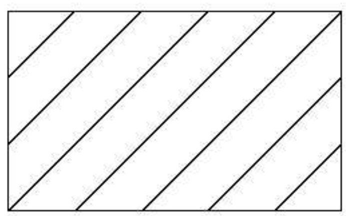
The standard table of knots and links only classifies prime knots and links, which makes the problem seem much easier. However, before it makes sense to only classify prime knots, it needs to be verified that composite knots are in some sense uniquely determined by their prime factors. Note that given any knot, one can factor it recursively into smaller and smaller components, until the resulting components are prime. Schubert's paper "Die eindeutige Zerlegbarkeit eines Knoten in Primeknoten" *S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl* 3 (1949) was the first to demonstrate that if decomposes a knot into prime factors, the factorization is unique up to their order.

The intention of this paper will be to demonstrate the existence of prime knots and that the factorization of knots into prime factors is unique up to the order of the operation. To show existence, we will use a knot's *genus,* an additive knot invariant. Although there are shorter ways to show the existence of prime knots, the use of the genus will allow us to show that the unknot is not the connected sum of two knots. This argument is what gives hope that there might be a unique factorization into prime components because no component of a knot can be "cancel" another component of the knot. Then, we present an argument similar to the one Schubert gave to show the uniqueness of the prime factorization.

Existence of prime knots:

The quickest way to show that there are prime knots is to show that any knot that can be embedded into the torus is prime. The way to do this is to first homotope a knot embedded into the torus into a standard form with straight lines. (Here the torus is represented as a rectangle with opposite sides identified.)

1)  becomes 2) 

and finally we get: 3) 

Notice that no matter how we try to intersect the torus with a sphere, we cannot obtain a factorization into two prime components. A sphere whose intersection with the torus looks like a circle (on the diagram above) with two intersection points with the knot can only determine the unknot. A sphere whose intersection looks like two vertical or horizontal lines cannot intersect the knot at exactly two points. Hence we cannot get a valid factorization into two nontrivial components.

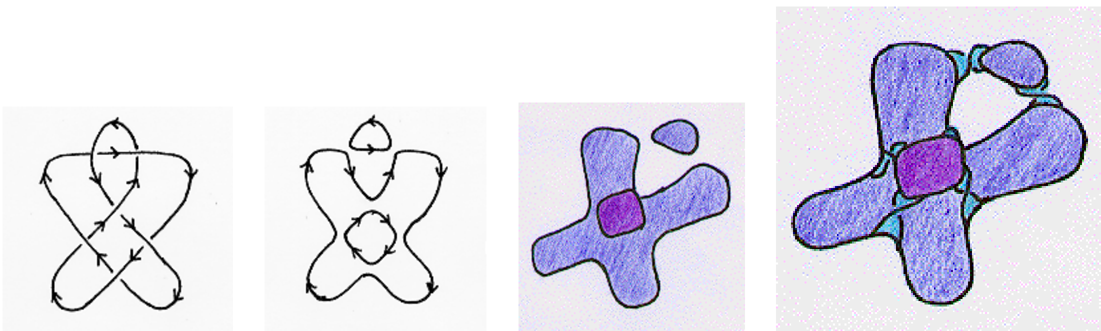
This short argument tells us that looking at the structure of (compact) surfaces that a knot can be embedded into might tell us something about the structure of the knot. Before considering surfaces that a given knot can be embedded into, let's consider surfaces that have the knot as a boundary.

Definition: *A* <u>*Seifert surface*</u> S_K *of a knot K is an orientable surface (in* $\mathbf{S}^3$*) that has boundary K.*

Seifert developed the following algorithm that constructs the diagram of a Seifert surface from the diagram of a knot:

1) Given a projection of a knot, orient it, and resolve all crossings, by joining the incoming segment from one strand to the outgoing component of the strand that crosses it. This will result in a collection of circular components called *Seifert circles*.
2) Now consider each Seifert circle as the boundary of a disk at a different level in 3-space.
3) Finally, reconnect the disks where the crossings were with a strip, giving it a half-twist. We will refer to these strips as *bridges*. The resulting surface will have the knot as its boundary.

Example:



Now let's verify that the surface that we have just constructed is orientable. Note that we can two-color the surface across any bridge, whether the Seifert circles that are being connected are nested or not. It is quite easy to see that if the two circles that are being connected are nested they have the same orientation and if they are not nested they have opposite orientations. Assign a consistent coloring to the two sides of the surface in the following way:

	Top	Bottom
If the Seifert circles run clockwise:	dotted	lined
If the Seifert circles run counterclockwise:	dotted	lined

This diagram shows that the 2-coloring of the diagram that we defined above remains consistent across any bridge, and therefore the entire surface can be 2-colored consistently. Hence it is orientable.

This algorithm is simple to implement, but provides a very important result: any knot is the boundary of some orientable surface. Now, by compactifying this orientable surface, we obtain an orientable compact surface in which the knot is embedded. This means that it will make sense to define knot invariants that are related to the surfaces in which a knot can be embedded.

Definitions: *The* <u>*associated compact surface*</u> $\hat{S}_K$ *of an orientable surface* S_K *is obtained by "sewing" a disk along each boundary component. (This is unique up to homotopy).*

Informally speaking, compact surfaces can be classified up to homotopy by the number of "holes" that they have. These "holes" are holes in the sense of a torus or a coffee cup, not puncture holes in a 2-dimensional surface.

The <u>*genus*</u> *of a compact surface is the number of "holes" that it has.*

The <u>*genus*</u> *of a knot K is the minimal genus over all associated compact surfaces of all Seifert surfaces of K.*

<u>Remark</u>: It is quite possible that the associated compact surface of minimal genus is cannot be obtained by the algorithm above.

In order to compute the genus of a surface, we'll use the *Euler Characteristic* of a surface S, which we'll denote by $\chi(S)$. To obtain the Euler Characteristic, we first need to *triangulate* our surface.

Recall that a triangulation is a division of our surface into 2-simpleces (=triangles) in such a way that the only intersections are 1-simpleces (=edges) shared by at most two 2-simpleces. Less formally, we make a model of our surface that is made up of triangles that are glued along the full length of their edges, and at most two triangles share an edge.

For a triangulation of the surface S, let V_S = #vertices, E_S = #edges, F_S = #faces.

Then
$$\chi(S) = V_S - E_S + F_S \qquad (1)$$

Now, let's relate the genus of a surface to its Euler Characteristic. We do this inductively by beginning with a sphere (genus 0) and increasing its genus by adding one torus at a time.

First, compute the Euler Characteristic of the connected sum of two triangulated surfaces. The connected sum of surfaces works exactly like the connected sum of knots. Remove a disk from each of the surfaces, and connect the boundaries. In order to simplify matters, the disk that we choose to remove from each surface will be one of the triangles from the triangulation. This will result in a triangulation of the new surface. We will lose three vertices, three edges and two faces from the earlier triangulations.

Thus:
$$\begin{aligned}\chi(S_1\#S_2) &= (V_1 + V_2 - 3) - (E_1 + E_2 - 3) + (F_1 + F_2 - 2)\\ &= \chi(S_1) + \chi(S_2) - 2 \qquad (2)\end{aligned}$$

Now, do induction on the genus g of a surface to obtain the following relation between the genus and the Euler characteristic of a Surface:

$$\chi(S) = 2 - 2g \Leftrightarrow g(S) = 1 - \chi(S)/2 \qquad (3)$$

To avoid having to triangulate compact surfaces, notice that triangulating the compact surface $\hat{S}_K$ is the same as triangulating the Seifert surface S_K and then triangulating the disk. Triangulate the disk in the following way. Add one vertex and then connect it to each of the vertices that lie along the knot. We add one vertex, n edges and n faces, and therefore:

$$\chi(\hat{S}_K) = \chi(S_K) + (1 - n + n) \qquad (4)$$

and then using (3)
$$g(\hat{S}_K) = 1 - \frac{\chi(S_K)+1}{2} \qquad (5)$$

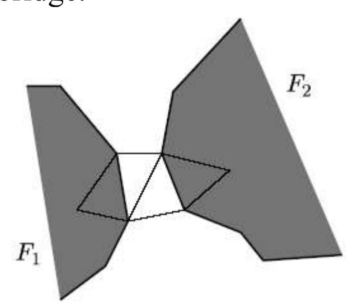
Since often we are considering surfaces that were obtained using Seifert's algorithm, the following relation will save us from having to triangulate surfaces once we've already gone through the algorithm.

<u>Proposition</u>: *Let s be the number of Seifert circles (as defined in the algorithm) and c be the number of crossings of a projection of a knot K. Then:*

$$g(\hat{S}_K) = 1 - \frac{s-c+1}{2} \qquad (6)$$

<u>Proof</u>: Choose a triangulation of the surface S_K obtained from the algorithm that has the following properties:

-There is one vertex in the middle of each Seifert circle, and n vertices on the Seifert circle, and n edges connecting the middle vertex to the others.
-There are vertices at each corner of each bridge, and one edge along the diagonal, as in the following diagram. The dark regions are Seifert circles and the light region is a bridge.



With this triangulation, each Seifert circle contributes n+1 vertices, 2n edges and n faces. The total contribution of each Seifert circle to $\chi(S_K)$ is: $(n+1) - (2n) + n = 1$.
Each bridge will add no extra vertices, 3 extra edges and two extra faces. The total contribution of a bridge to $\chi(S_K)$ is: $0 - 3 + 2 = -1$.
Putting these two facts together we get:

$$\chi(S_K) = 1 \cdot s + (-1) \cdot c = s - c,$$

and by (5) $$g(\hat{S}_K) = 1 - \frac{s-c+1}{2} \quad . \qquad (7)$$

□

We now have the machinery to prove the main result that will allow us to draw conclusions about knots based on their genus: the genus is an additive invariant under the connected sum operation.

<u>Theorem</u>: *For two knots K_1 and K_2*

$$g(K_1\# K_2) = g(K_1) + g(K_2) \qquad (8)$$

To prove this result, first construct minimal Seifert surfaces $\hat{S}_{K1}$ and $\hat{S}_K$ for each of the knots K_1 and K_2. Triangulate them in such a way that at least two vertices appear "close enough" along each knot, connected by a small piece of the knot. Then take the direct sum of the surfaces using triangles that include the chosen vertices, and that lie in S_{K1} and S_{K2}.

The important thing to notice is that the resulting surface is a Seifert surface for $K_1\#K_2$. We have taken the direct sum of the two knots lying on the surface, by cutting them between the two chosen vertices, and identifying the boundaries. The surfaces S_{K1} and S_{K2} lie on the same side of the resulting knot. They have been connected along the two edges of each triangle that were inside the surfaces, and now form a single surface. The resulting surface that has $K_1\#K_2$ as a boundary. The disks that we sewed on to make

$\hat{S}_{K1}$ and $\hat{S}_{K2}$ have been joined to make one big disk. This big disk is "sewed" along the boundary of $K_1\#K_2$ and forms an associated compact surface of $K_1\#K_2$.

Now, by (2) $\quad \chi(\hat{S}_{K1}\# \hat{S}_{K2}) = \chi(\hat{S}_{K1}) + \chi(\hat{S}_{K2}) -2$.

Using (3):

$$g(\hat{S}_{K1} \text{¿} \hat{S}_{K2}) = 1 - \frac{\chi(\hat{S}_{K1}) + \chi(\hat{S}_{K2}) - 2}{2}$$

$$= 1 - \frac{\chi(\hat{S}_{K1})}{2} + 1 - \frac{\chi(\hat{S}_{K2})}{2}$$

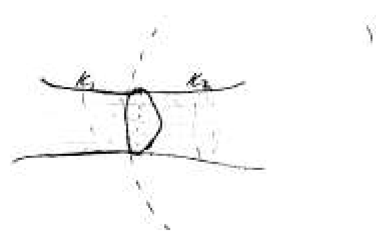
$$= g(\hat{S}_{K1}) + g(\hat{S}_{K2}).$$

Since $\hat{S}_{K1}\# \hat{S}_{K2}$ is an associated compact surface to a Seifert surface for $K_1\# K_2$,

$$g(K_1\# K_2) \leq g(K_1) + g(K_2).$$

To establish the other inequality, build a Seifert surface $\hat{S}_{(K1\# K2)}$ of minimal genus for $K_1\# K_2$. Then select a sphere that will separate the knot into its factors. It can be selected so that all of its intersections with $\hat{S}_{(K1\# K2)}$ are 1-dimensional. Triangulate the surface and the interiors of the intersections between the surface and the dissecting sphere. Then split the surface into $\hat{S}_1$ and $\hat{S}_2$, "capping off" all holes with the interiors of the intersections. The resulting surfaces will be associated compact surfaces to Seifert surfaces of the knots K_1 and K_2. The disk is split into two disks, and the Seifert surface into two Seifert surfaces.

We again examine the resulting triangulation. Consider the case when the intersection with the sphere is a circle. All other cases (multiple cirles, that might be nested) involve a similar algebraic manipulation of the Euler characteristic.



$$\chi(\hat{S}_1) + \chi(\hat{S}_2) = (V_{(S1\# S2)} + n + 2) - (E_{(S1\# S2)} + 3n) + (F_{(S1\# S2)} + 2n)$$

$$= \chi(\hat{S}_{(K1\#K2)}) + 2$$

Using (3):

$$g(\hat{S}_1)+g(\hat{S}_2)=1-\frac{\chi(\hat{S}_1)}{2}+1-\frac{\chi(\hat{S}_2)}{2}$$

$$\text{¿}\,2-\frac{\chi(\hat{S}_1)+\chi(\hat{S}_2)}{2}$$

$$\text{¿}\,2-\frac{\chi(\hat{S}_{(K1\#K2)})+2}{2}$$

$$\text{¿}\,1-\frac{\chi(\hat{S}_{(K1\#K2)})}{2}=g(S_1\#S_2)$$

Hence $g(K_1)+g(K_2)\leq g(K_1\#\,K_2)$,

and the argument is complete.

<u>Proposition</u>: *A knot has genus 0 if and only if it is the unknot:*

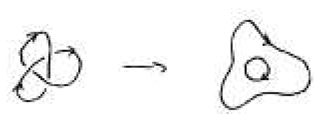
The unknot's Seifert surface is a disk, therefore its associated compact surface is a sphere. As we've seen above: $g = 0$ (by (7)).
On the other hand, if $g = 0$, the knot's associated compact surface is a sphere. Therefore the knot can be embedded in a sphere. Cut the sphere along the knot, and that will result in two disks. The disk must be the knot's minimal Seifert surface, whose boundary, K is the unknot.

□

<u>Corollary</u>: *Knots do not have inverses under the direct sum operation.*

Any nontrivial knot has strictly positive genus. Therefore connected sum of two nontrivial knots will also have strictly positive genus, and cannot be the unknot.

<u>Example</u>: Let's compute the genus of the trefoil. To do this, begin Seifert's algorithm:



Then, by (7), we only need to count the number of Seifert circles and the number of intersections in the diagram:

$$g(\hat{S}_K) = 1- (2-3 +1)/2 = 1$$

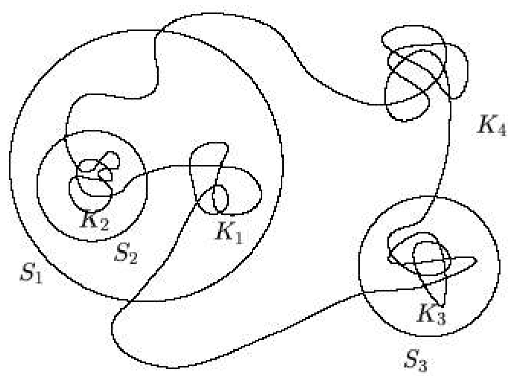
We know that the trefoil is nontrivial, and by the remark above, its genus is strictly greater than 0. The computation above bounds the genus of the trefoil by 1, therefore the genus of the trefoil must be 1.

That means that if the trefoil is the connected sum of two knots, one must have genus 1 and the other genus 0. This is a trivial factorization, and we conclude that **the trefoil is prime**. We have now shown the existence of prime knots.

Uniqueness

The following definitions do not introduce any new ideas. They will serve as shorthand as we build the uniqueness argument.

Definition: *A* <u>*dissecting sphere system*</u> ***S*** *of a knot K is a collection of nonintersecting spheres that "assign one prime factor per region" of 3-space. When the knot is factored along these spheres, we get a factorization into prime components.*



We'll call the region that a sphere s_i determines int(s_i) This is the region of 3-space between the sphere s_i and all spheres inside of s_i. On the diagram above int(s_1) is the region that determines K_1.

We say that ***S** ~ **S'*** *if* ***S*** *and* ***S'*** *determine the same factorization of K.*

Let's now attack the main result:

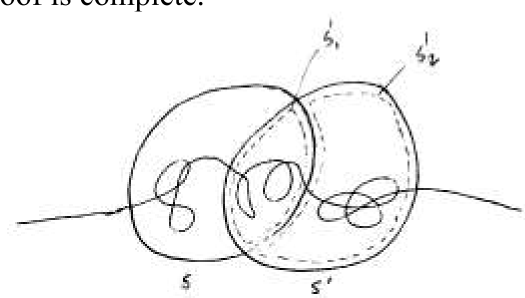
<u>*Theorem*</u>*:* *Any two factorizations of a knot K determines the same prime knots.*

We argue that for any **S** and **S'** that dissect a knot K into prime factors, **S** ~ **S'**. We will only sketch the proof here. (To make the proof rigorous, one needs to introduce lots of notation to describe the rather intuitive idea of having one prime factor in each "region on 3-space", where a region of 3-space is a region in between a sphere and all spheres that are inside it.) The argument is built by doing induction on the number of components in the intersection of the two dissecting sphere systems.

If n = 0: To argue here we'll do induction on m + m', the sum of the number of spheres in the two dissecting sphere systems.
The base case is trivial.
For the inductive step, select a sphere s_i that is innermost with respect to **S**. It must be outermost with respect to some s'_j in S', and hence determine the same prime component. Put **S''** = **S'** – s'_i s'_j. Then, **S'~S''**. Now factor out the prime component inside s'_i. The remaining knot must have (m + m' – 1) components since the s'_i does not determine the unknot. Apply the induction hypothesis to each of those pieces, and we get the same prime factors on each of them, and hence **S''~S**. We conclude that **S~S''~S'**.

If n > 0: Assume the result is true for (n – 1) components. Reduce the number of components as follows:

Choose a pair of intersecting spheres s and s' in **S** and **S'** respectively. Consider the regions int(s) int(s') and int(s')\int(s). They cannot both contain knotted components since the knot in s' is prime. They cannot both determine unknots since otherwise the resulting knot would be the unknot. Replace s' with ŝ'(one of the dotted spheres on the diagram below), the sphere that contains a knotted component. We now have reduced the number of intersections by one, and by induction, the proof is complete.



Conclusion: The unique factorization of knots into prime components is an important step in understanding knots because it is a drastic reduction. Instead of having to look for invariants of all knots, it suffices to look at invariants of prime components and how those invariants behave under the connected sum operation. It allows knot theorists to focus their attention on smaller components that make up knots. The proof also simplifies the way in which one can look at knots: it doesn't matter in which order one tries to break a knot into prime components, the resulting factors will be the same.

The construction of Seifert surfaces was a useful tool in this argument. Even though it was a very geometric construction and is difficult to implement, the genus has been used to distinguish between knots that had not previously been distinguished, which makes it a useful invariant, as in the Kinoshita-Terasaka mutants [see Adams]. An important question about Seifert's algorithm still remains unsolved: which knots have a projection such that when Seifert's algorithm is applied to it yields a surface of minimal genus? All that is known is that alternating projections of alternating knots yield a Seifert surface of minimal genus [see Gabai]

Bibliography:

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This textbook contains many details about Seifert surfaces and genus of knots.

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