Lecture 7.

(Finish lecture 6).

Recall **Thm. 1.** The breadth of the bracket of any reduced alternating knot diagram with c crossings is $4c$.

Proof of Thm. 1.

Let σ_A be the state with all splicings being A-splicings (as before).

The highest power in $\langle K / \sigma_A \rangle$ is

$$c + 2 \cdot \|\sigma_A\|$$

(c – number of crossings,
$\|\sigma_A\| = \#(\text{components in } \sigma_A) - 1$).

Let σ_1 be a state where all splicings are of type A, except for one, which is of type B.

Then the highest power in $\langle K / \sigma_1 \rangle$ is

$$(c-2) + 2(\|\sigma_1\| - 1)$$

By lemma 1,

$$(c-2) + 2(\|\sigma_1\| - 1) < c + 2\|\sigma_A\|.$$

By lemma 2, no state other then σ_1 can contribute a degree higher then that of σ_1.

Thus, max degree $= c + 2\|\sigma_A\|$.

Reversing the roles of A and B-splicings,

min degree $= -(c + 2\|\sigma_B\|)$.

The breadth is

$$2c + 2(\|\sigma_A\| + \|\sigma_B\|).$$

But components of σ_A are boundaries of white regions,
— " — " σ_B — " — " — black regions.

Thus, $\|\sigma_A\| + \|\sigma_B\| = \#\text{comp.}(\sigma_A) + \#\text{comp.}(\sigma_B) - 2 = c. \Rightarrow$ The breadth $= 4c$

Proof of Thm. 2

We need to see what changes if we consider an arbitrary diagram (not necessarily reduced alternating).

In Lemma 1: there is no strict drop of degree from σ_A to σ_1.
Lemma 2: valid.

Thus, maximal power (which may or may not occur) is still

$$c + 2\|\sigma_A\|.$$

Thus, breadth $\leq 2c + 2\left(\|\sigma_A\| + \|\sigma_B\|\right)$.

However, we still need an estimate on $\|\sigma_A\| + \|\sigma_B\|$ for an arbitrary diagram.

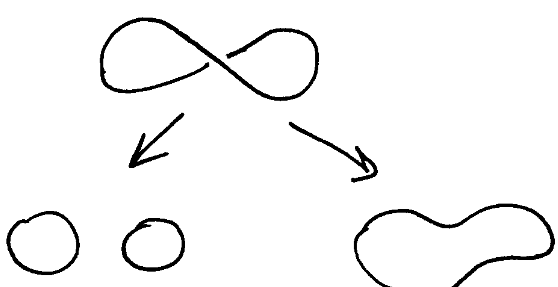
<u>Lemma 3.</u> For a state σ, let σ^* be the <u>dual state</u>, obtained by exchanging all As and Bs. Then
~~Let σ be a~~

$$\|\sigma\| + \|\sigma^*\| \leq c,$$

where c is the number of crossings in the original diagram.

<u>Proof</u>. (Induction on c):

$c = 1$:



σ $\qquad$ σ^*

$\|\sigma\| = 1$
$\|\sigma^*\| = 0.$

The lemma is true.

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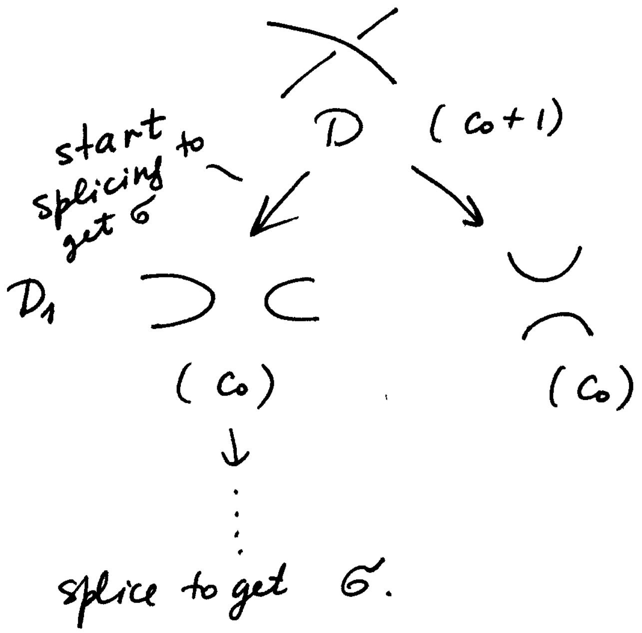
Assume that the lemma is true for all $c \le c_0$.
Let D be a diagram with (c_0+1)-crossings.



– 1^{st} step.

– one of thes diagrams is connected. E.g., the left one.

Let $\sigma^*_{(D)}$ be the dual of σ w.r.to D.
Let $\sigma^*(D_1)$ be the dual of σ w.r.to D_1.

Then $\sigma^*(D_1)$ differs from $\sigma^*(D)$ only at the original crossing.

Thus,
$$\| \sigma^*(D_1) \| = \| \sigma^*(D) \| \pm 1. \qquad (*)$$

By induction hypothesis we have
$$\| \sigma(D_1) \| + \| \sigma^*(D_1) \| \le c$$

Substituting $(*)$, we obtain:
$$\| \sigma \| + \| \sigma^* \| \pm 1 \le c$$

which implies
$$\| \sigma \| + \| \sigma^* \| \le c + 1.$$

∎

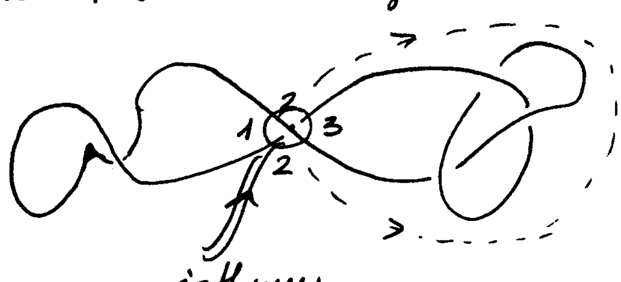
The Jones polynomial and alternating links

Tait's conjecture: alternating diagrams are minimal (any knot represented by an alternating diagram cannot be represented by any other diagram with fewer crossings).
(No proof was known for about 100 years, until the discovery of the Jones polynomial).

Def. A diagram is connected if the underlying projection is connected subset of the plane. (Clearly, any knot diagram is connected).
A diagram divides the plane into several regions.

Lemma. If the diagram is connected, all regions are homeomorphic to disc.
And $\#(\text{regions}) = \#(\text{crossings}) + 2$.

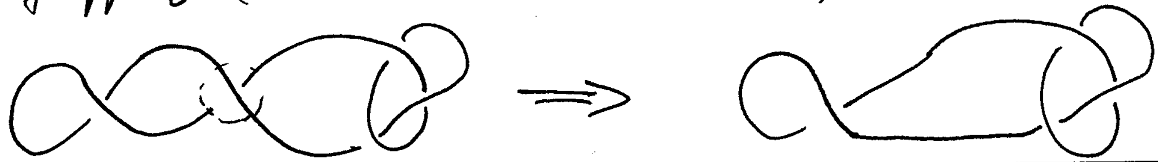
Def An isthmus is a crossing at which there are less then 4 distinct regions:



same region

(Isthmus is like a bridge between 2 separate diagrams).
One can destroy an isthmus by flipping (the half of diagram): (flip the part left to the isthmus)

Ex.



– no isthmus

<u>Def</u> A diagram is <u>reduced</u> if there are no isthmi.
(Any diagram can be made reduced by flipping a half of diagram several times).

<u>Def</u> The <u>Breadth</u> of a Laurant polynomial is the difference between the highest and lowest powers of the variable.

<u>Ex.</u> Breadth $(t^3 + t - 17t^{-4}) = 7.$

<u>Claim.</u> The breadth of the bracket of the knot is an invariant (need only consider R1).

<u>Theorem 1.</u>
The breadth of the bracket polynomial of a reduced alternating knot diagram with c crossings is exactly 4c.

<u>Theorem 2.</u>
The breadth of the bracket of any knot diagram with c crossings is $\leq 4c$.

<u>Corrolary</u> (Proof of Tait's conjecture)
Any reduced alternating diagram is minimal.

<u>Proof of Corrolary</u>
Let D be reduced alternating diagram with c crossin
Then breadth of the bracket of D = 4c.
But Breadth is a knot invariant. Thus by Thm there can't be any diagrams of the same knot with fewer than c crossings

This is equivalent to:

① Any non-trivial reduced alternating knot diagram represents a non-trivial knot.

② All reduced alternating diagrams of the same knot have the same # of crossings.

<u>Proofs of Thms 1 & 2:</u>

Recall that $$\langle K \rangle = \sum_{\sigma} \langle K \mid \sigma \rangle,$$

where σ is a state of K.

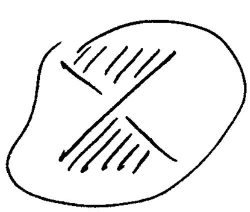
If σ_A is a state with only A-resolutions and
σ_B —//— //— B-resolutions,
$\overset{\shortparallel}{A^{-1}}$

then
σ_A contributes highest positive power of A,
σ_B —//— lowest (negative) power of A.

Let $|\sigma|$ be (the number of connected components (or, loops) in σ) -1

<u>Lemma 1.</u>
For a reduced alternating diagram with σ_A, σ_B as above,
$$|\sigma_A| > |\sigma_1|$$ for any state σ_1 which has exactly 1 B-splitting.

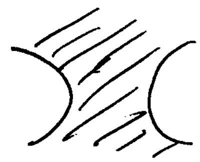
<u>Proof</u>



– Color (by black & white) regions at each crossing.

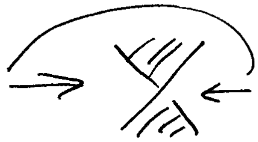
Split D to σ_A. Then loops of σ_A are boundaries of ~~black~~ white regions:

A-splicing with 2-~~coloring~~ shading



For σ_1, one crossing will be different:

Different regions



Since the diagram was reduced, this vertex is not an isthmus. Thus two white regions in the left diagram are different. On the right they are connected.

Thus
$$|\sigma_1| = |\sigma_A| - 1.$$

<u>Lemma 2.</u>

D - diagram with c crossings.

σ_k - any state with k B-splicings (and $c-k$ A-splicings)

Let $\sigma_A = \sigma_0, \sigma_1, \dots, \sigma_{k-1}, \sigma_k$ be a chain of states (obtained from each other) and having i B-splicings for σ_i.

Then the max power in $\langle D | \sigma_{i+1} \rangle \leq \langle D | \sigma_i \rangle$ $\forall i$.

<u>Proof</u>

$$\langle D | \sigma_i \rangle = (-1) \cdot A^{c-2j} \cdot (A^2 + A^{-2})^{|\sigma_i|}$$
$$\langle D | \sigma_{i+1} \rangle = (-1) A^{c-2(j+1)} (A^2 + A^{-2})^{|\sigma_{i+1}|}$$

Note
$$|\sigma_{i+1}| = |\sigma_i| \pm 1$$

Power of $(A^2 + A^{-2})$ increases at most by 1. But $\Sigma\sigma$ dec. by 2.