Lecture 6.

The Jones Polynomial
(from the bracket polynomial)

Def. The Jones polynomial $V_K(t)$ is a Laurant polynomial in t assigned to an oriented link K so that the following properties are satisfied:

(1) $V_K(t)$ is invariant under ambient isotopy;

(2) $V_{\bigcirc}(t) = 1$

(3) $t^{-1} \cdot V_{\times} - t \cdot V_{\times} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right) \cdot V_{\asymp}$.

Q: 1) Existence?
2) Well-defined?

Define $\boxed{V_K(t) \doteq \mathcal{L}_K(t^{-1/4})}$, where $\mathcal{L}_K$ is the normalized bracket polynomial.

Thm. $V_K(t)$ defined as above satisfies (1)–(3).

Proof (1), (2) follow from the corresponding properties of $\mathcal{L}_K(t)$. Need to check (3):

$$\langle \times \rangle = A \cdot \langle \asymp \rangle + B \langle)(\rangle .$$

$$\langle \times \rangle = B \cdot \langle \asymp \rangle + A \langle \asymp \rangle .$$

$$B^{-1} \langle \times \rangle - A^{-1} \langle \times \rangle = \left(\frac{A}{B} - \frac{B}{A}\right) \cdot \langle \asymp \rangle .$$

$$\boxed{A \langle \times \rangle - A^{-1} \langle \times \rangle = (A^2 - A^{-2}) \langle \asymp \rangle^{(*)}}$$ (compare with (3)).

Let $w = w(\ )$.

Then $w(\ ) = w+1$ and $w(\ ) = w-1$.

Let $\alpha = -A^3$.

Then, multiplying (*) by α^{-w}, we get

$$A \langle\ \rangle \alpha^{-w} - A^{-1} \langle\ \rangle \alpha^{-w} = (A^2 - A^{-2}) \langle\ \rangle \alpha^{-w}.$$

$$A \cdot \alpha \langle\ \rangle \alpha^{-(w+1)} - A^{-1} \cdot \alpha^{-1} \cdot \langle\ \rangle \alpha^{-(w-1)} =$$

$$= (A^2 - A^{-2}) \langle\ \rangle \alpha^{-w}.$$

$$-A^4 \mathcal{L} + A^{-4} \mathcal{L} = (A^2 - A^{-2}) \mathcal{L}.$$

Putting $A = t^{-1/4}$, we get property (3).

<u>The reversing property of the Jones polynomial.</u>

<u>Thm.</u> Let K be a link, let $K_1, \dots, K_n$ be its components. Let K' be the link obtained from K by reversing the direction of one component (e.g., K_1). Let $\lambda = lk(K_1, K - K_1)$ be the total linking number of K_1 with the rest of K.

Then

$$V_{K'}(t) = t^{-3\lambda} V_K(t).$$

Proof.

The effect of reversing of direction of K_1 on the writhe of the link:

E.g.

$= \Sigma$

$$lk(K_1, K-K_1) = \frac{1}{2} \underbrace{\sum_{c \in \underbrace{K_1 \cap K-K_1}_{\text{crossings of diagrams}}} \text{sign}(c)}_{}$$

$$w(K) = w(K-K_1) + w(K_1) + \Sigma$$

$$w(K') = w(K-K_1) + w(K_1) - \Sigma$$

$$\Rightarrow w(K') = w(K) - 2\Sigma = w(K) - 4\, lk(K_1, K-K_1).$$

Thus, $\quad w(K') = w(K) - 4\, lk(K_1, K-K_1)$.

$$\boxed{w(K') = w(K) - 4\lambda}.$$

Thus,

$$L_{K'}(A) = (-A^3)^{-w(K')} \langle K' \rangle =$$

$$= (-A^3)^{-w(K')} \langle K \rangle =$$

$$= (-A^3)^{-w(K)+4\lambda} \langle K \rangle =$$

$$= (-A^3)^{4\lambda} \cdot L_K(A).$$

Thus, $V_{K'}(t) = L_{K'}(t^{-1/4}) = t^{-3\lambda} L_K(t^{-1/4}) = t^{-3\lambda} V_K(t)$.

∎

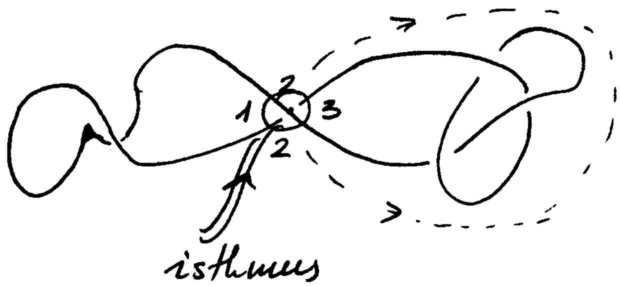
The Jones polynomial and alternating links

<u>Tait's conjecture:</u> alternating diagrams are minimal
(any knot represented by an alternating diagram cannot be represented by any other diagram with fewer crossings).
(No proof was known for about 100 years, until the discovery of the Jones polynomial).

<u>Def.</u> A diagram is <u>connected</u> if the underlying projection is connected subset of the plane. (Clearly, any knot diagram is connected).
A diagram divides the plane into several <u>regions</u>.

<u>Lemma.</u> If the diagram is connected, all regions are homeomorphic to disc.
And $\#(\text{regions}) = \#(\text{crossings}) + 2$.

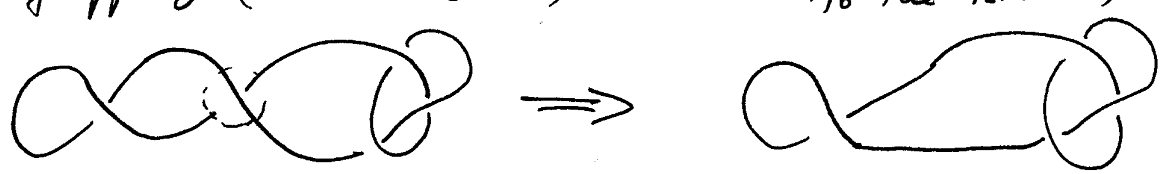
<u>Def</u> An <u>isthmus</u> is a crossing at which there are less then 4 distict regions:



same region

(Isthmus is like a bridge between 2 separate diagrams).
One can destroy an isthmus by flipping (the half of diagram). (flip the part left to the isthmus)

Ex.



– no isthmus.

<u>Def</u> A diagram is <u>reduced</u> if there are no isthmi.
(Any diagram can be made reduced by flipping a half of diagram several times).

<u>Def</u> The <u>breadth</u> of a Laurant polynomial is the difference between the highest and lowest powers of the variable.

<u>Ex.</u> Breadth $(t^3 + t - 17t^{-4}) = 7$.

<u>Claim.</u> The breadth of the bracket of the knot is an invariant (need only consider R1).

<u>Theorem 1.</u>

The breadth of the bracket polynomial of a reduced alternating knot diagram with c crossings is exactly 4c.

<u>Theorem 2.</u>

The breadth of the bracket of any knot diagram with c crossings is $\leq 4c$.

<u>Corrolary</u> (Proof of Tait's conjecture)

Any reduced alternating diagram is minimal.

<u>Proof of Corrolary</u>

Let D be reduced alternating diagram with c crossings.
Then breadth of the bracket of D = 4c.
But Breadth is a knot invariant. Thus by Thm.2 there can't be any diagrams of the same knot with fewer than c crossings

This is equivalent to:

(1). Any non-trivial reduced alternating knot diagram represents a non-trivial knot.

(2). All reduced alternating diagrams of the same knot have the same # of crossings.

<u>Proofs of Thms 1 & 2:</u>

Recall that
$$\langle K\rangle = \sum_{\sigma} \langle K \mid \sigma\rangle,$$
where σ is a state of K.

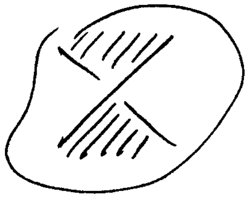
If σ_A is a state with only A-resolutions and
σ_B —"— "— B-resolutions, ($= A^{-1}$)

then
σ_A contributes highest positive power of A,
σ_B —"— lowest (negative) power of A.

Let $|\sigma|$ be (the number of connected components in σ) $- 1$
(or, loops)

<u>Lemma 1.</u>
For a reduced alternating diagram D with σ_A, σ_B as above,
$$|\sigma_A| > |\sigma_1|$$
for any state σ_1 which has exactly 1 B-splitting.

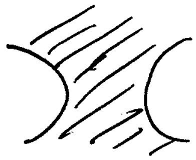
<u>Proof</u>



Color (by black & white) regions at each crossing.

Split D to σ_A. Then loops of σ_A are boundaries of ~~black~~ white regions:

A-splicing with 2-~~coloring~~ shading



For σ_1, one crossing will be different:

Different regions

Since the diagram was reduced, this vertex is not an isthmus. Thus two white regions in the left diagram are different. On the right they are connected.

Thus
$$|\sigma_1| = |\sigma_A| - 1.$$

<u>Lemma 2.</u>

D - diagram with c crossings.

σ_k - any state with k B-splicings (and $c-k$ A-splicings)

Let $\sigma_A = \sigma_0, \sigma_1, \dots, \sigma_{k-1}, \sigma_k$ be a chain of states (obtained from each other) and having i B-splicings for σ_i.

Then the max power in $\langle D | \sigma_{i+1} \rangle \leq \langle D | \sigma_i \rangle$ $\forall i$.

<u>Proof</u>

$$\langle D | \sigma_i \rangle = (-1) A^{c-2j} \cdot (A^2 + A^{-2})^{|\sigma_i|}$$
$$\langle D | \sigma_{i+1} \rangle = (-1) A^{c-2(j+1)} (A^2 + A^{-2})^{|\sigma_{i+1}|}$$

Note
$$|\sigma_{i+1}| = |\sigma_i| \pm 1$$

Power of $(A^2 + A^{-2})$ increases at most by 1. But $\Sigma\sigma$ dec. by 2.