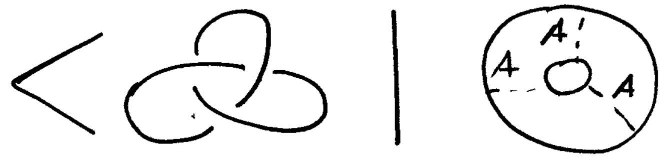
Lecture 5.

States and the bracket polynomial.

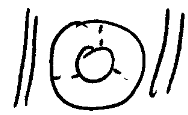
Let K be a knot and σ be its state.
Let $\langle K | \sigma \rangle$ denote the product of the labels attached to σ.

Ex. 

$= A^3$.

(Labels are determined from the structure relative to K).

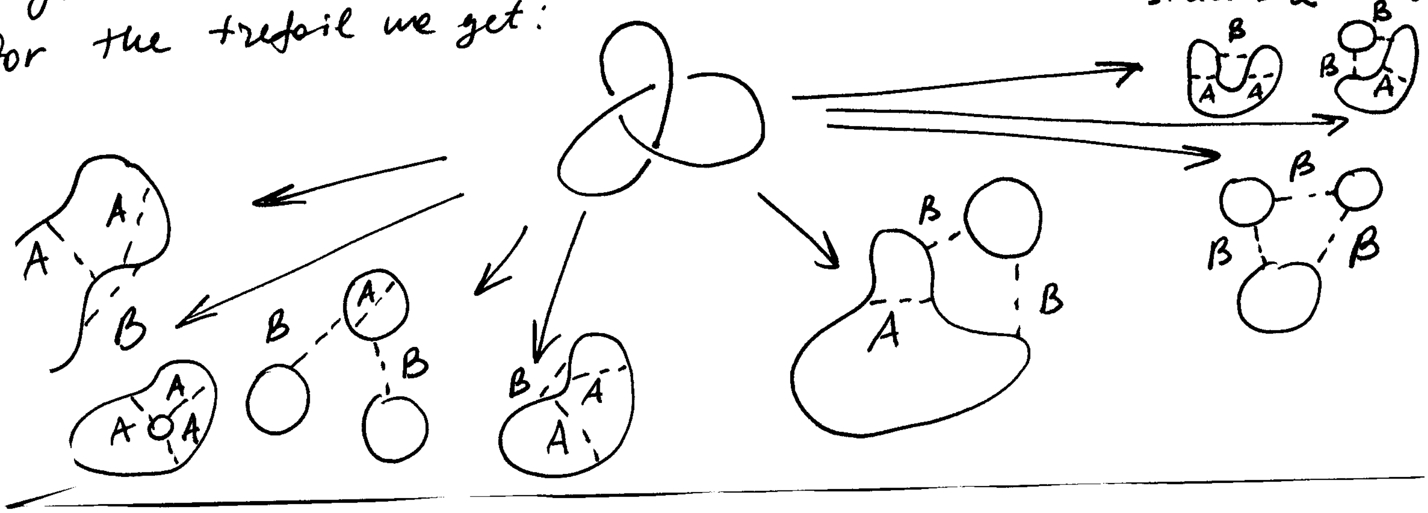
Let
$\|\sigma\|$ be the number of components -1.

Ex.  $= 1$.

Def. The <u>bracket polynomial</u> is a polynomial in 3 variables A, B, d (commuting)

$$\langle K \rangle = \langle K \rangle (A, B, d) = \sum_{\sigma} \langle K | \sigma \rangle \cdot d^{\|\sigma\|}$$

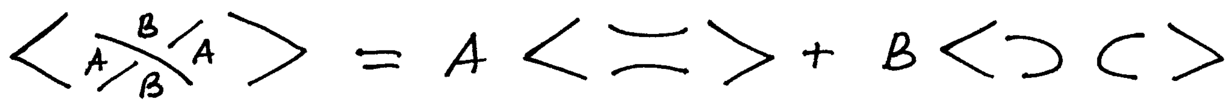
<u>Example</u>. Can compute the bracket polynomial from state decomposition.
E.g., one can show that (we started this procedure last time)
for the trefoil we get: (The number of states is $2^{\#\text{crossing}}$)



Thus, the bracket polynomial of the trefoil is:

$$\langle K\rangle = \underline{A^2B} + A^3d + AB^2d + \underline{A^2B} + AB^2d + B^3d^2 +$$
$$+ \underline{A^2B} + AB^2d =$$
$$= 3A^2B + (A^3 + AB^2 + AB^2 + AB^2)d + B^3d^2 =$$
$$= 3A^2B + (A^2 + 3B^2)\cdot A\cdot d + B^3\cdot d^2.$$

<u>Lemma.</u> (A recursive property of the bracket polynomial).



<u>Rmk.</u> Regard any diagram in the lemma as a part of the larger diagram.

<u>Proof</u> Let K be a knot and K_A be the knot obtained by an A-splicing at some crossing; let K_B be the knot obtained by B-splicing of K at the same crossing. Then

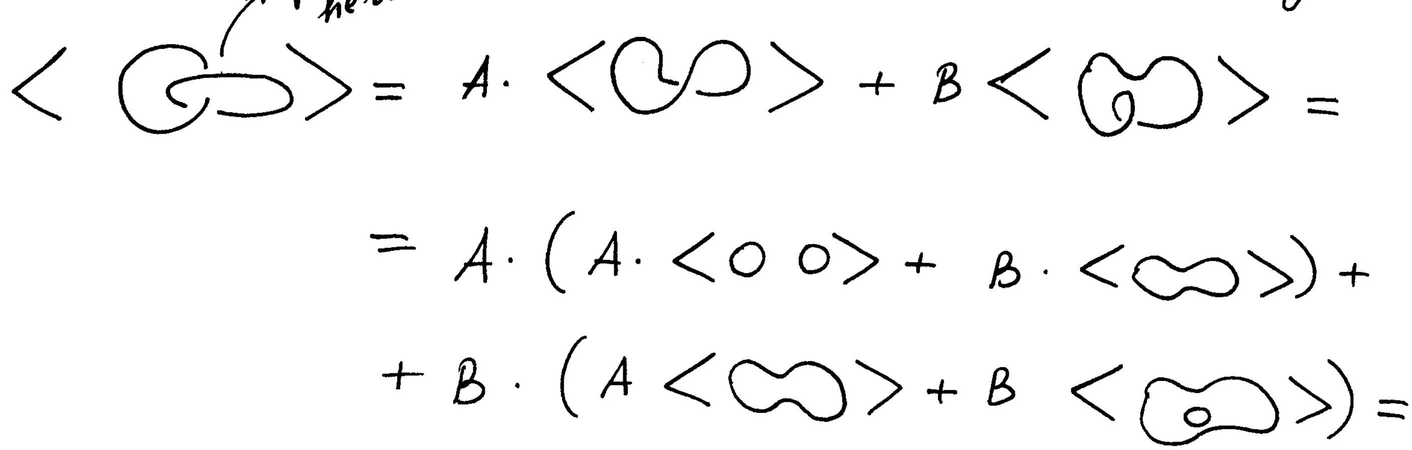
$$\text{States of } K = (\text{States of } K_A) \cup (\text{States of } K_B)$$

By definition of the bracket polynomial, it follows that

$$\langle K\rangle = A\cdot\langle K_A\rangle + B\cdot\langle K_B\rangle,$$

which is the statement of the lemma.

Ex. Using this recursion to compute the bracket polynomial

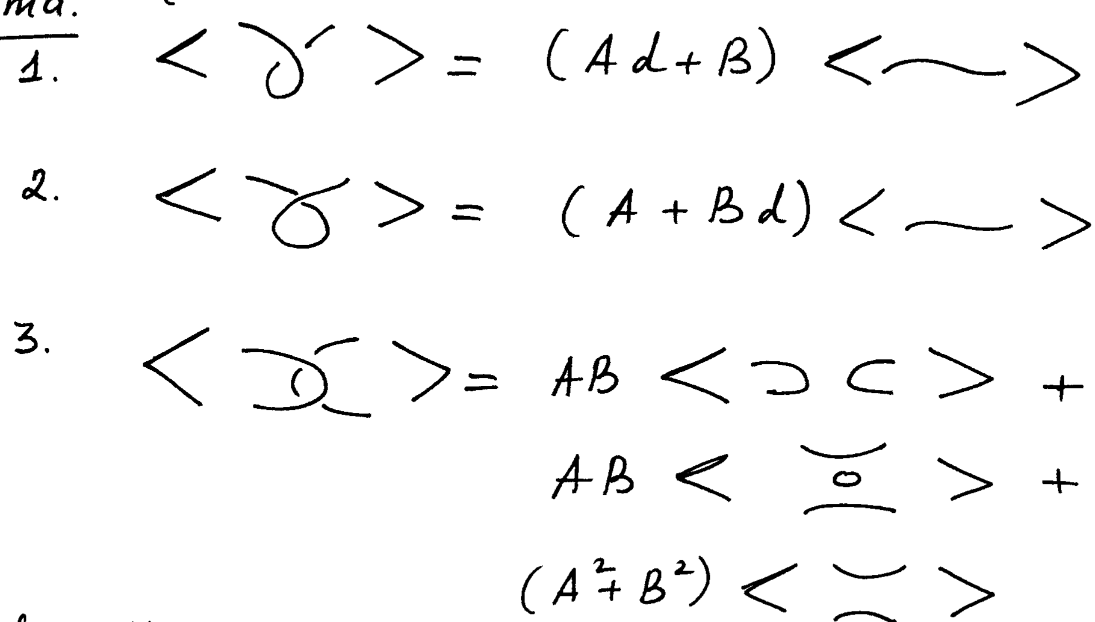


$$= A^2 d + AB + AB + Bd =$$

$$= 2AB + (A^2+B)d.$$

Q: What is the behavior of the bracket polynomial under the Reidemeister moves?

Lemma. (Behavior under R1 and R2):



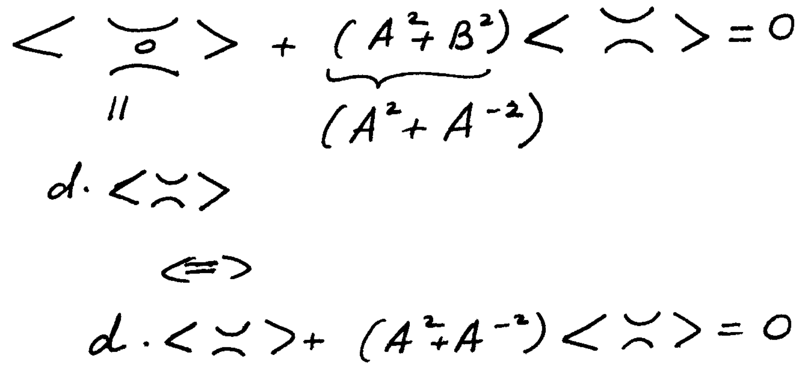
Proof. Homework.

We want to add conditions on A, B, d so that they are preserved under the move ~~[illegible]~~ R2:

(*)

which requires

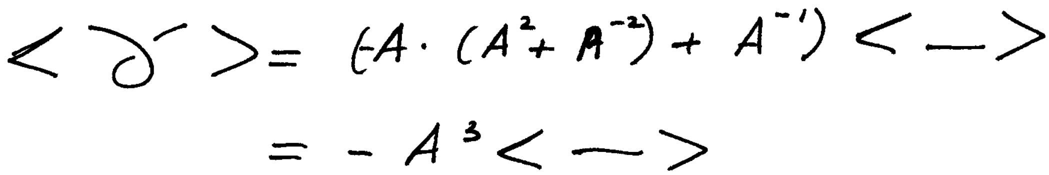
(1) $\boxed{AB = 1}$ and ~~A~~



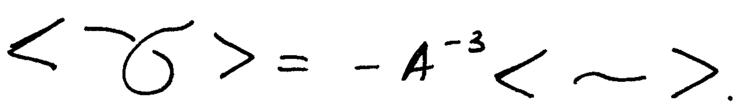
(2)

Assume $\boxed{d = -(A^2+A^{-2})}$

With these conditions, in addition to (*) we get



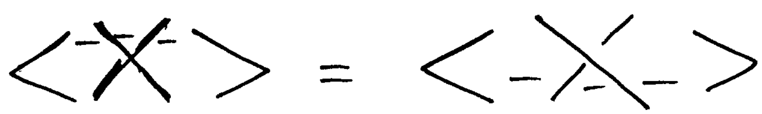
Similarly,

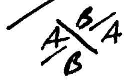


So, by requiring (1) and (2), we can garantee that the bracket is invariant under R2.

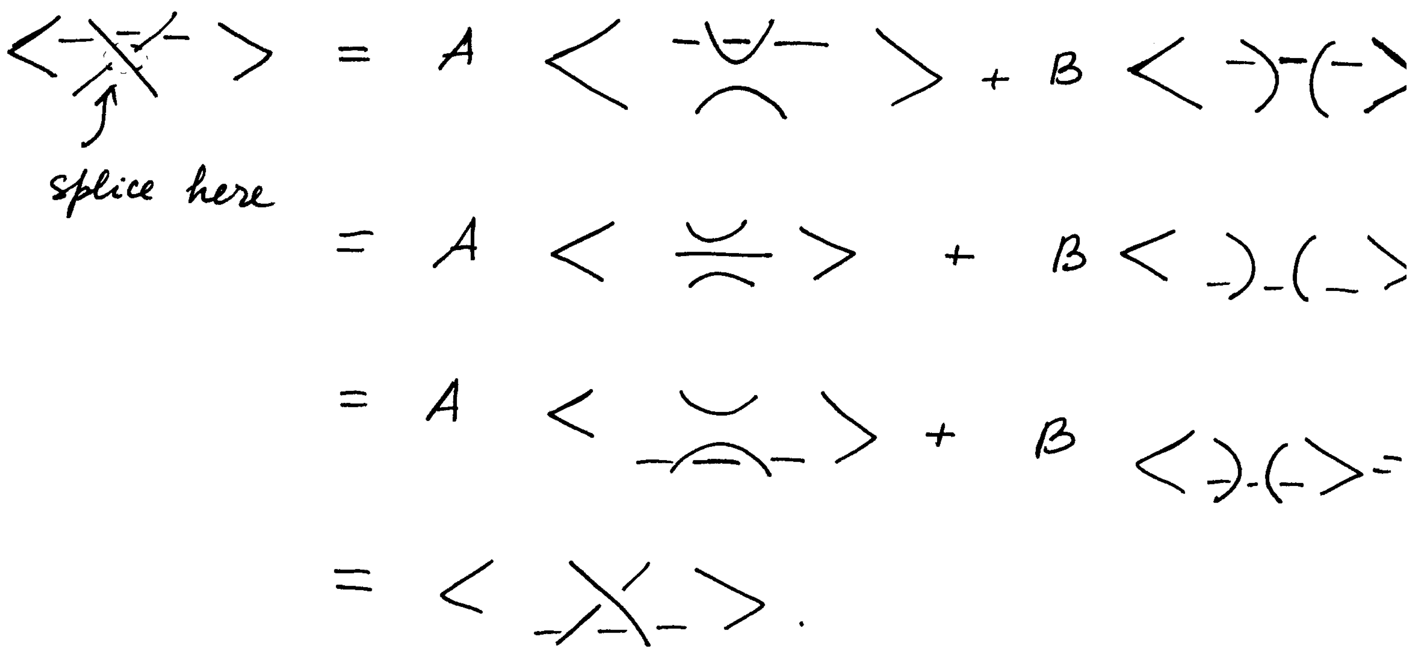
It turns out that the same conditions ensure that $\langle\ \rangle$ is invariant under R3:

<u>Lemma.</u> If $B = A^{-1}$, $d = -(A^2+A^{-2})$, then



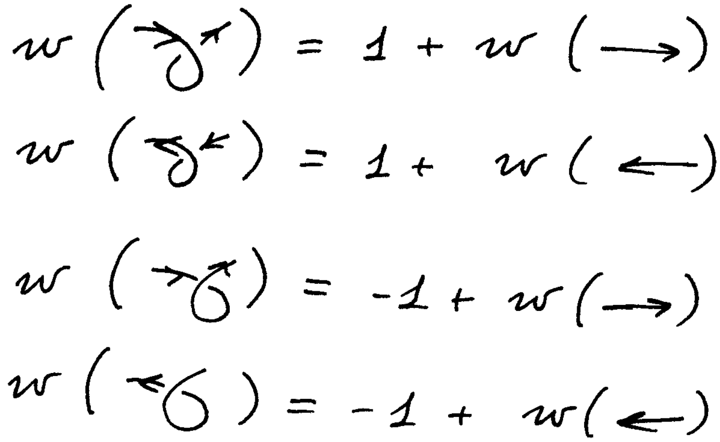


Proof



To make $\langle \ \rangle$ invariant under R1, need to recall the <u>writhe</u> of a knot:

Recall that $w(K)$ is invariant ~~and~~ under R2, R3 and changes under R1 as follows:



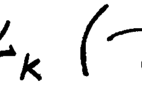
Define the normalized bracket

$$L_K = (-A^3)^{-w(K)} \cdot \langle K \rangle .$$

Thm. The normalized bracket L_K is invariant under the ambient isotopy.

Proof $w(K)$ and $\langle K \rangle$ are inv. under R2, R3 (regular isotopy).

Under R1:

L_K () $= (-A^3)^{-w(}$  $^{)}$ $\langle$  $\rangle =$

$$= (-A^3)^{-(1+w(\rightarrow))} \cdot (-A^3) \langle - \rangle$$

$$= (-A^3)^{-w(\rightarrow)} \cdot \langle - \rangle = L_K(\rightarrow).$$

Bracket polynomials of the mirror images.

K - knot; K^* - mirror image of K (switch all the crossings on a diagram).

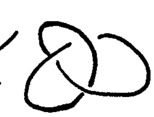
Crossing change in a diagram $\Leftrightarrow$ switch roles of A and A^{-1} ($= B$).

Then $\langle K^* \rangle (A) = \langle K \rangle (A^{-1})$

$$L_{K^*}(A) = L_K(A^{-1}).$$

(L_K is very close to the original Jones polynomial!)

Thus, if $L_K(A) \neq L_K(A^{-1})$, then K is not ambient isotopic to K^* (chirality!)

Ex. Trefoil $w(T) = 3$ $\langle$  $\rangle$

$$\langle K \rangle = 3A^2B + (A^2 + 3B^2) \cdot A \cdot d + B^3 d^2 =$$

$$= 3A + (A^2 + 3A^{-2}) \cdot A \cdot (-1) \cdot (A^2 + A^{-2}) + A^{-3} \cdot (A^2 + A^{-2})^2 =$$

$$= \ldots = -A^5 - A^{-3} + A^{-7}.$$

L  $= (-A^3)^{-3} \langle$  $\rangle = -A^{-9}(-A^5 - A^{-3} + A^{-7}) =$

$= A^{-4} + A^{-12} - A^{-16}.$

L  $= A^4 + A^{12} - A^{16}.$

Since $L_K \neq L_{K^*}$, the trefoil is not equivalent to its mirror image.

<u>Ex.</u> Torus links of type $(2, n)$

K_1 K_2 K_3 K_4

Find a recursive