Lecture 4.

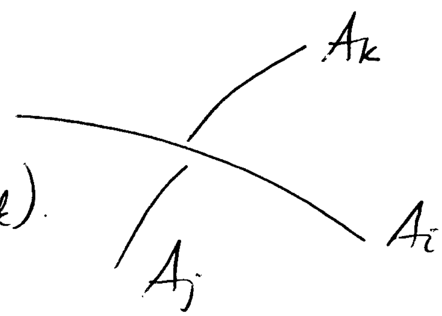
p-colourings.

Assume that p is a prime.
The colours $\{0, 1, \ldots, p-1\}$ form a field $\mathbb{F}_p$.
(Then $T_p(D)$, the set of p-colourings of D, is a vector space over $\mathbb{F}_p$.
If p is not prime, $T_p(D)$ would only be a module over the ring $\mathbb{F}_p$..)

<u>Def.</u> For a prime p, $T_p(D)$ - the set of colourings of arcs of a diagram D.
At a crossing with arcs A_i, A_j, A_k, the condition is:

$$x_j + x_k = 2x_i \mod p$$

(where A_i goes over A_j and A_k).



$$T_p(D) = \{(x_1, \ldots, x_k) \in \mathbb{F}_p^k : \; x_j + x_k = 2x_i \mod p \text{ at crossing with arcs } A_i \text{ going over } A_j, A_k\}.$$

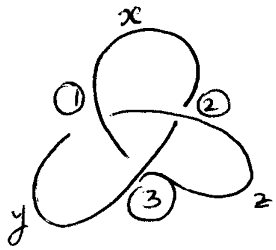
(If $p = 3$:
$x_j + x_k = 2x_i \mod 3$
$2x_i = -x_i \mod 3$
Thus, $x_i + x_j + x_k = 0 \mod 3$ - the old condition)

$T_p(D)$ is the vector space (over $\mathbb{F}_p$) of solutions of linear homogeneous equations.

$$\tau_p(D) = p^{\dim T_p(D)}$$

Examples.

(1).



Let $y=0$.

$x,y,z \in \{0,\dots,p-1\}$.

(1): $y+z=2x \mod p$

or, $\boxed{z=2x \mod p}$

(2): $\boxed{x = 2z \mod p}$

(3): $\boxed{x+z=0 \mod p}$

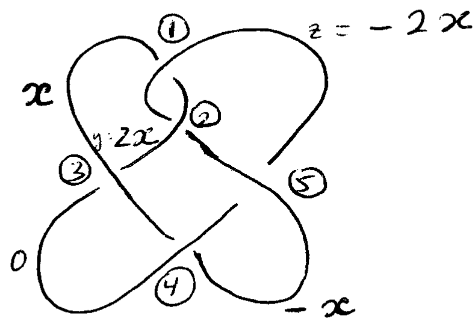
Thus,
$$\left.\begin{aligned} z &= -x \mod p \\ -x &= 2x \mod p \\ x &= -2\cancel{x} \mod p \end{aligned}\right\} \Rightarrow 3x = 0 \mod p.$$

Thus, $3x = p\cdot k$

Assume $x \neq 0 \mod p$ (otherwise we get a constant colouring)
Then p is divisable by 3.
So, 3_1 is colourable mod p iff $3 \mid p$.

(2).

5_2.



(3). $2x = y \mod p$
$\Rightarrow \underline{y = 2x}$.

(1). ~~$2x+x = 2z \mod p$~~

~~$z =$~~

(4).

(2). $2\cdot(2x) = -x + (-2x) \mod p$ $\qquad 4x = -3x \mod p$

(5). ✓

(1). $-2x\cdot 2 = 2x + x \mod p$ $\qquad -4x = 3x \mod p$.

$\Rightarrow$ 5_2 has a coloring mod p iff p is divisible by 7.

(Thus, 5_2 is not equivalent to 3_1).

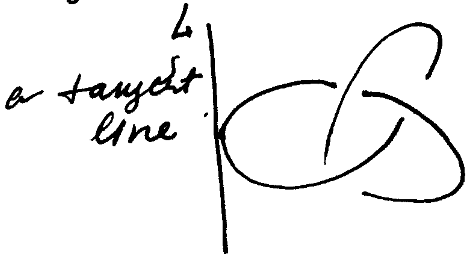
The unknotting number.

Def. The <u>unknotting number</u> $u(K)$ is the minimum over all diagrams D of K, of the minimal number of crossing changes required to turn D into a diagram of the unknot.

Lemma. Any knot diagram can be changed to a diagram of the unknot by switching some crossings.

Proof. See homework 2.

Hint: take a knot K and its diagram D obtained by a projection onto a plane.



Diagram

Let (x,y) be the coordinates in the plane of the projection.
Let $t \mapsto (x(t), y(t), z(t))$ be a parametrization of the knot.
Consider the knot K' obtained by gluing
$t \mapsto (x(t), y(t), t)$ to a vertical segment connecting its endpoints, $(p,0)$ and $(p,1)$.

Then K' has the same projection and is the unknot

Thm. for any knot, $u(K) \leq \frac{1}{2} c(K)$.

Proof Apply such a procedure to a diagram with a minimal crossing number, $c(K)$. In this way, we use at most $c(K)$ crossing changes.
If $u(K) \geq \frac{c(K)}{2}$, change K to the unknot K'' with z-coordinate $1-t$. (Change the crossings that we didn't change the first time). Thus,

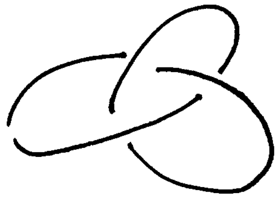
$$\boxed{u(K) \leq \frac{c(K)}{2}}.$$

Alternating knots.

Alternating diagram – a diagram on which types of crossings alternate (over, under, over, ...) if we go around the diagram starting with an arbitrary pt.

Alternating knot – a knot which has an alternating diagram.

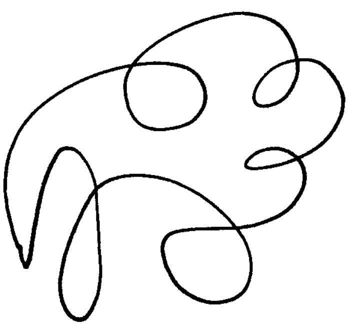
Ex. Trefoil.

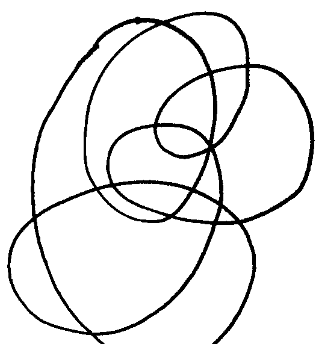


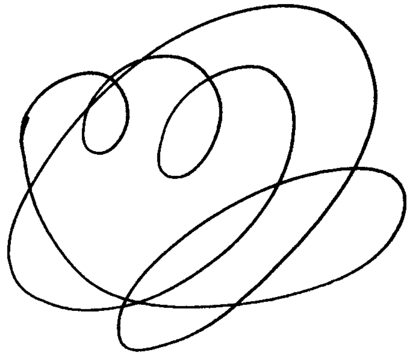
Hard research problem: Give an intrinsic 3-dim def. of an alternating knot (no reference to diagrams!)

A simple way to create a knot diagram:

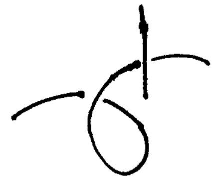
Start with an arbitrary projection:







Then resolve



etc.

Q: Why does this never leads to a contradiction?

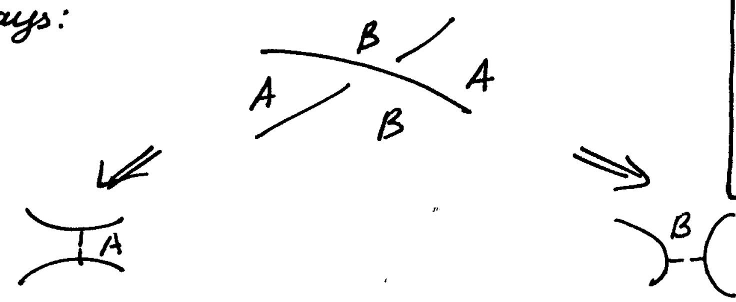
Question (Tait)

Is every alternating diagram minimal?
Does every non-trivial alternating diagram represent a non-trivial knot

(The answer is, basically, yes. The proof relies on the Jones polynomial).

States and the bracket polynomial.

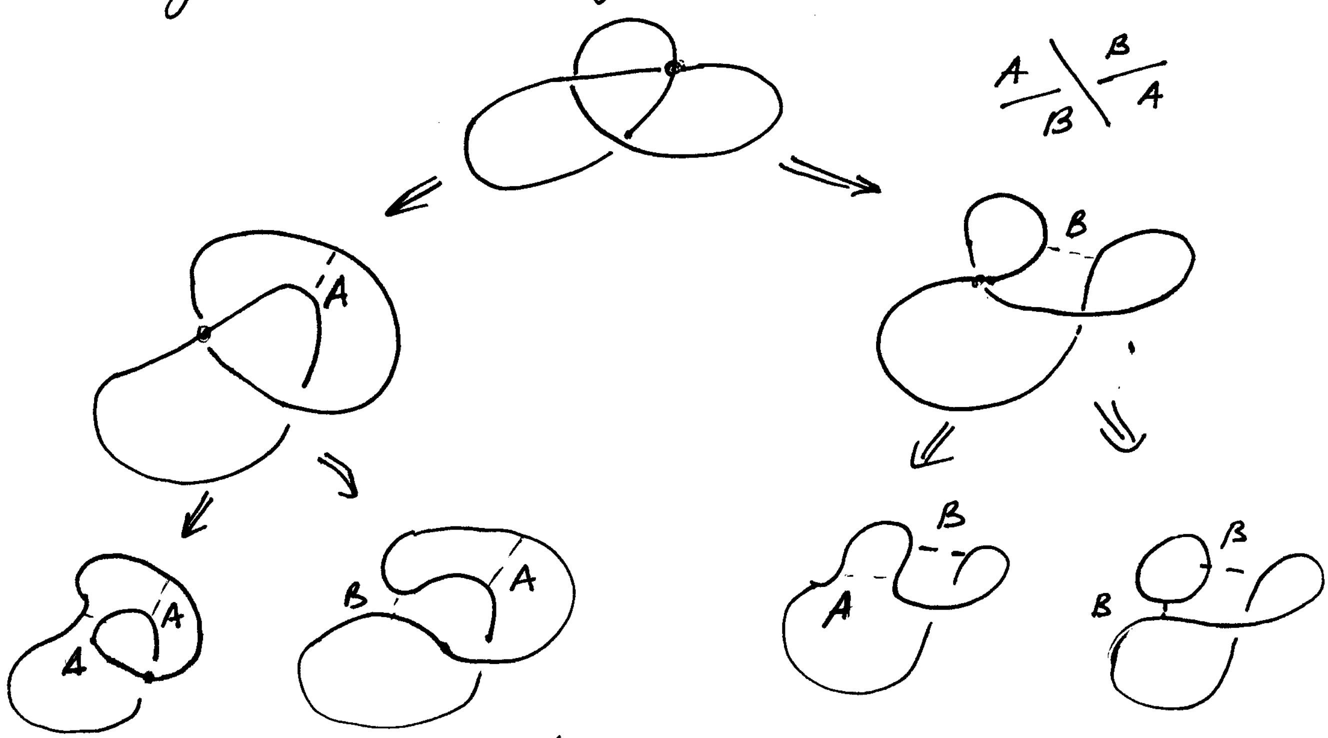
Splice a crossing in 2 ways:



A-split joins the regions A.

B-split joins the regions B.

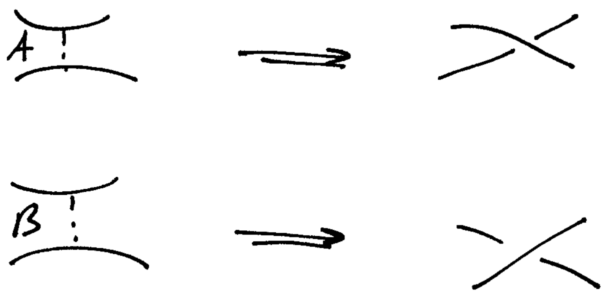
By splicing (resolving) the crossings, one can obtain a family of diagrams whose common ancestor is the original link diagram:

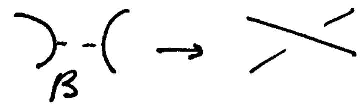


etc.

Eventually get diagrams of n-copies of unknot...

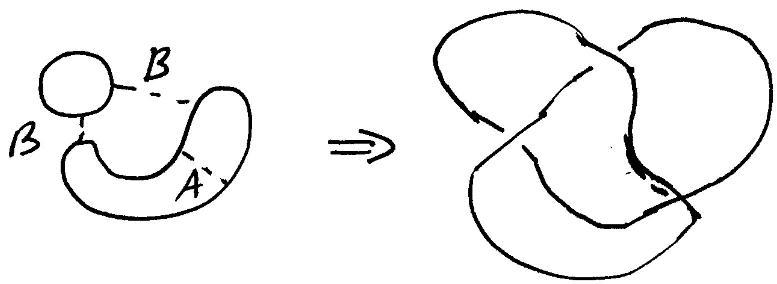
A split site labeled A or B can be <u>reconstructed</u> to form its ancestral crossing:





or:

<u>Ex.</u> Reconstruct from



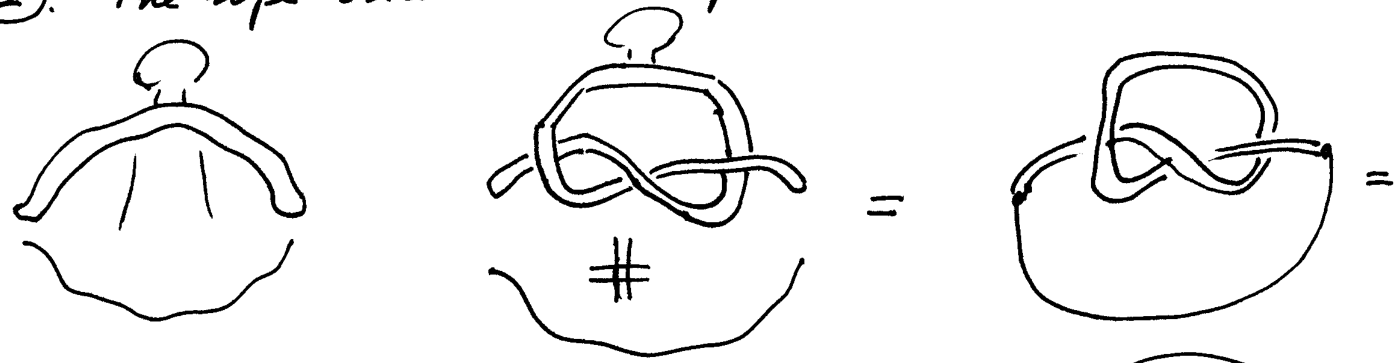
Final descendants are called <u>states.</u>

Any state (with labelings) can be used to reconstruct K.

We will construct invariants of knots and links by averaging over these states.

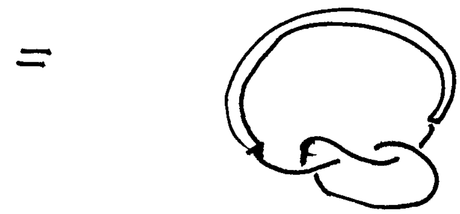
The fun part of the lecture.

① The rope trick and its explanation.



(Connected sum is commutative:

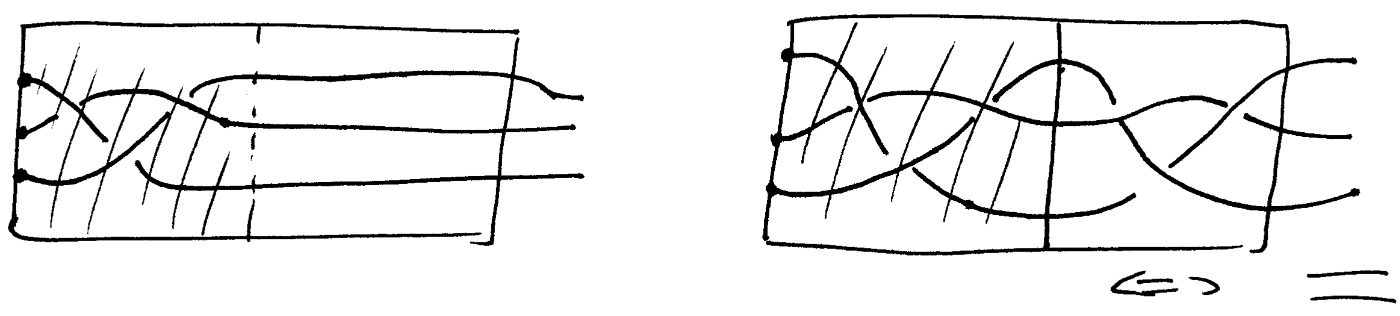
$K \# O = O \# K$).



② Unknotting vs. unbraiding.

No knot can be untied by adding another knot...

Not so with brading:



Braids form a group under connected sum.
(Knots only form a semi-group).