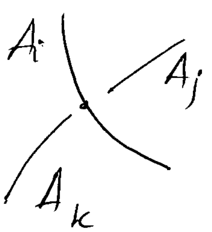
Lecture 3.

Finish lecture 2.

Then:

Consider the set of all assignments of colours $x_i \in \{0, 1, 2\}$ to arcs A_i.

When do we have an honest 3-coloring?



colors x_i, x_j, x_k must form one of the triples

$(0,0,0)$ $\quad$ $(0,1,2)$ & permutation
$(1,1,1)$
$(2,2,2)$

9 triples: they ~~satisfy~~ are exactly the solutions of

$$x_i + x_j + x_k = 0 \mod 3$$

(the equations has 9 solutions)

Can think of the colours as elements of the field $\mathbb{F}_3$ of 3 elements (addition, multiplication; both commutative + distributivity). 0, 1, 2

Then

$$T(D) = \{(x_1, \dots, x_k) \in \mathbb{F}_3^k : x_i + x_j + x_k = 0$$

at each crossing ~~with~~ of arcs $A_i, A_j, A_k\}$.

$T(D)$ - set of solutions of l homogeneous linear eq. in k unknowns over the field $\mathbb{F}_3$

<u>Thm</u> $T(D)$ is an $\mathbb{F}_3$-vector space.

Therefore, $\tau(D) = 3^{\dim T(D)}$ is a power of 3.

<u>Proof.</u> Solutions form a vector space, with the number of elements $3^{\dim T(D)}$.

<u>Computing $\tau(D)$:</u>

Let k be the number of arcs on a diagram — arcs: $(A_1, \dots, A_k)$

—‖— l —‖— —‖— crossings — crossings: $(C_1, \dots, C_l)$

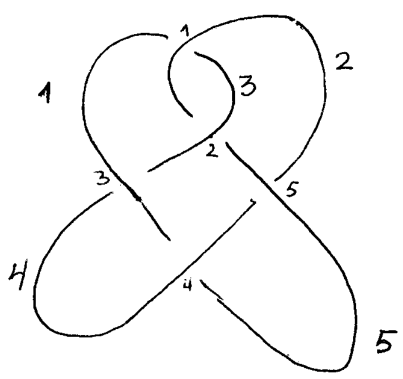
Associate to a diagram the matrix A of size $l \times k$ (over $\mathbb{F}_3$) encoding the l equations from crossings. We want to find the dimension of the space of solutions of $Ax = 0$.

Dimension = nullity of A

Compute by Gaussian elimination.

Note: everything is <u>mod 3</u> !

<u>Example.</u> 5_2:



5×5-matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Row reduction:

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 & -1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ & & & & \end{pmatrix}$$