1/ Bring colored chalk!

Lecture 2.

Announce Office Hours:
Try Th: 2-3; 4^{15}-4^{45}
T: 1:30-3

Knot Inv. - any function depending only on the equivalence class up to ambient isotopy.

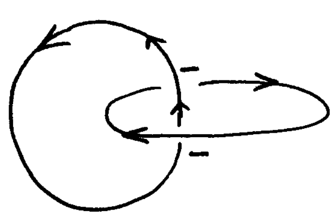
← ## The linking number.

Consider a link with two components, α and β.
Let $\alpha \sqcap \beta$ denote the set of crossings of α with β (not including self-crossings).
Then the *linking number* of α and β is

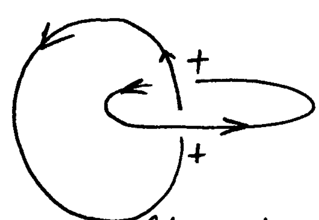
$$lk(\alpha,\beta) = \frac{1}{2} \sum_{c \in \alpha \sqcap \beta} \underbrace{\varepsilon(c)}_{\text{sign of crossing}}$$

Thm. lk is an invariant.
(Check how it behaves under R1, R2, R3).

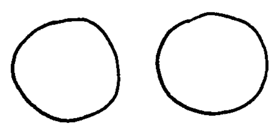
Ex. 1. The Hopf link: two distinct possible mutual orientations



$lk = -1$



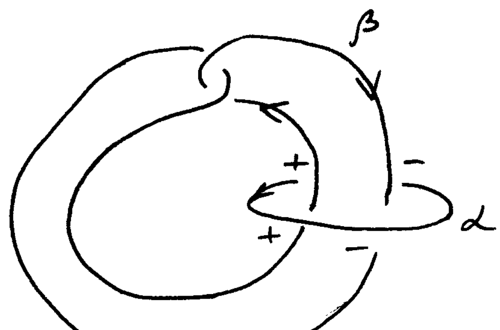
$lk = 1$



$lk = 0$

⟹ The Hopf link is not equivalent to 2-component unlink.

2. The White head link:



$$lk(\alpha,\beta) = \frac{1}{2}(1+1-1-1) = 0$$

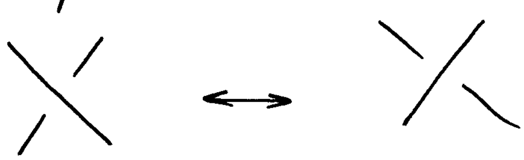
The linking number is 0, but, clearly, the diagram suggests that linking is non-trivial!

Def. The crossing number – minimal number of crossings that occurs in any diagram of the knot K. (So far, the only crossing number we can compute is that of the unknot)

Operations on knots:

Mirror image: for a knot K, its mirror image is obtained by reflecting it in a plane in $\mathbb{R}^3$. (All such reflections are equivalent!)

On a diagram, this amounts to to changing all crossings to the opposite ones. (This is just a reflection in the plane of the board:

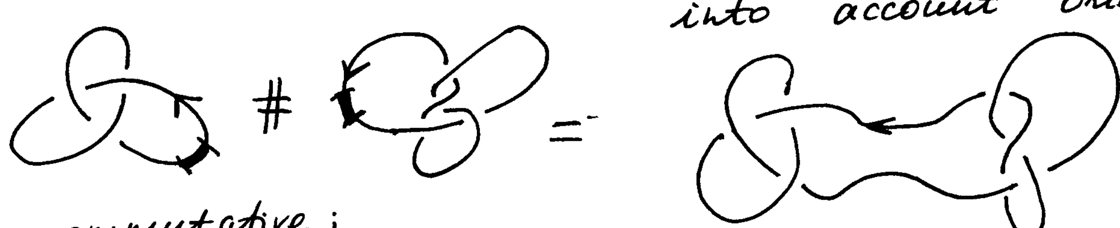


(We'll see that trefoil is not eq. to its mirror image)

Reverse: the reverse of an oriented knot is the same knot with opposite orientation.

Connected sum of oriented knots: K_1, K_2 – oriented knots

$K_1 \# K_2$ – remove an arc from each, connect the ends to get a single component taking into account orientation.



$\#$ is commutative;

$\bigcirc$ is the identity,

but there is no inverse. (will prove this later) $\Rightarrow$ knots form a semigr[oup] under connected sum.

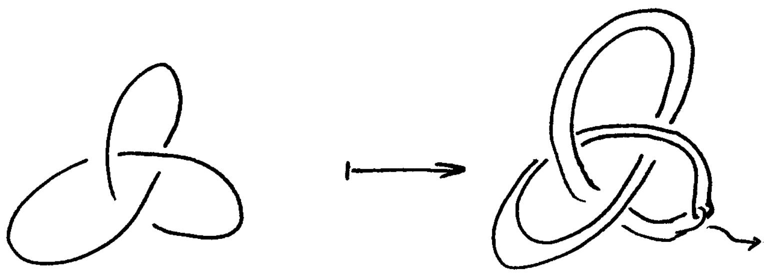
2/

Ex. 3. (The linking number of a Whitehead doubling of a knot).

(add to the end of page 1').

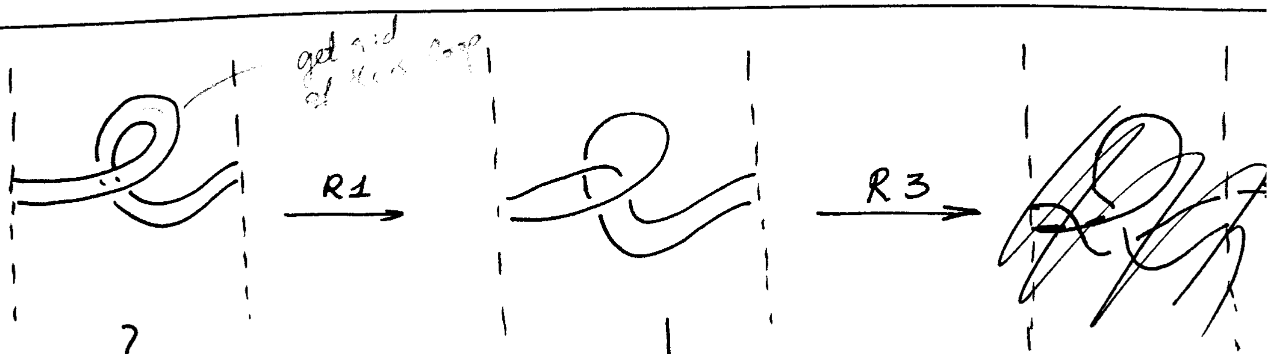
Whitehead double: • replace knot by two parallel copies (they can twist around each other in several different ways

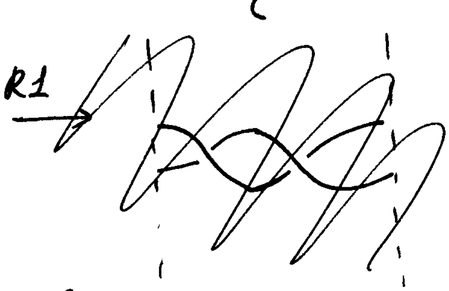
• add a "clasp" to join the two resulting components:



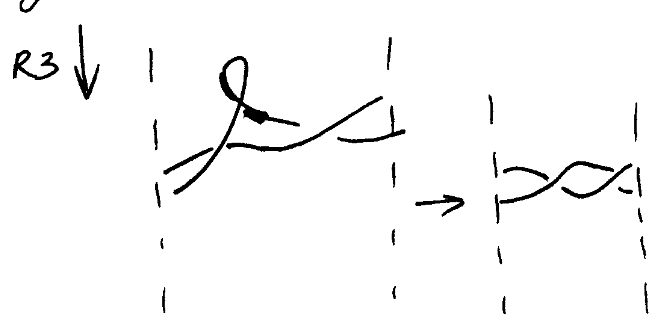
clasp.

Ex.

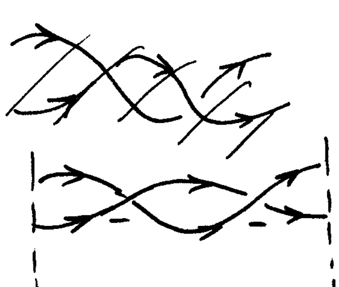




(This is similar to two arcs forming the edges of a belt)

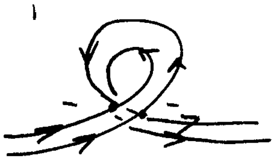


The linking numbers:

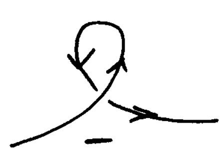


$lk = -1$

(Full negative twisting)



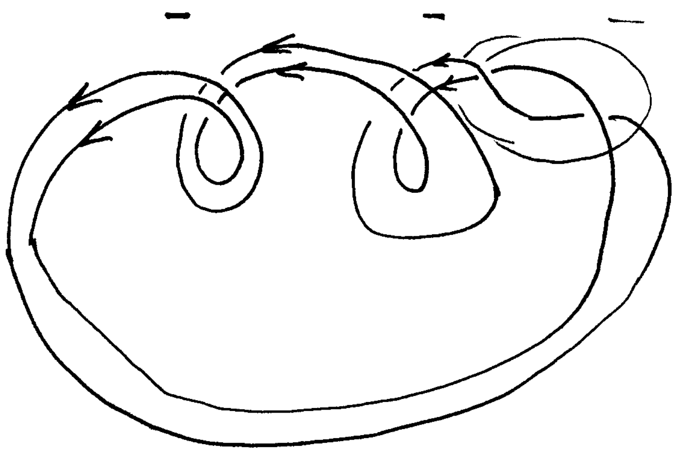
$lk = -1 \quad \Longleftrightarrow$



$w = -1$.

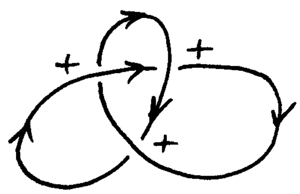
(One curl)

<u>Ex.</u>



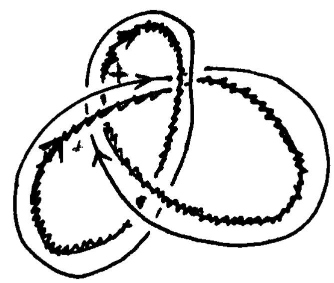
$lk = -3.$

<u>Ex.</u> Build a link by adding a parallel strand to a knot:

$K =$ 

$w(K) = 3$

form a link $\longmapsto$

 $= L$

$lk(L) = 3.$

If we add extra twisting, each full positive twist

contributes $+1$.

$T\Big($  $\Big) = 1$

<u>Thm.</u> (Whitehead) The linking number of parallel twisted strands is the sum of the writhing and the twisting:

not invariant

$$lk(L) = w(K) + T(L)$$

K is the knot associated to L. ; twisting of L

(Note: one can now easily make up "linked" links with linking n. = 0)

3-colourings.

The number of 3-colourings of a knot ~~is~~ a simple computable invariant, which is defined in a combinatorial way.

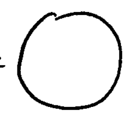
Choose 3 colors. ~~Colo~~ A 3-colouring of a link diagram is a choice of colours for each of the arcs so that

(∗) At each crossing, the three arcs that meet at the crossing are either all the same colour, or all three colours are used.

Let D be a diagram and $T(D)$ be the set of 3-colourings of this diagram.

Let $\tau(D)$ be the number of 3-colourings.

$\tau(D) \leq 3^k$, where k is the number of arcs.

Ex. ①. Standard diagram of unknot: $D=$ 

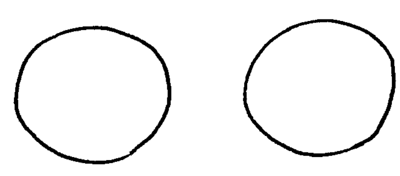
$\tau(D) = 3.$

②. Standard diagram of the trefoil:

$k = 3$

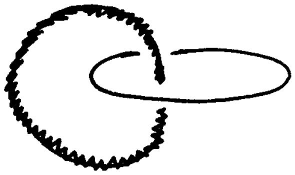
All three arcs meet at each of the three crossings.

$\tau(D) = 3\cdot 2 + 3 = 9.$

③. 2-component unlink: 

$\tau(D) = 3^2 = 9$

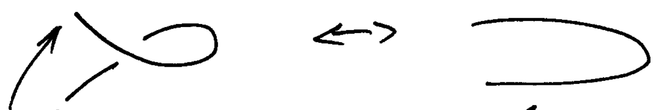
5/ ④. The Hopf link (standard diagram)

$D =$  $\tau(D) = 3.$

<u>Theorem.</u> The number of 3-colourings is a link invariant (i.e., is independent of the choice of a diagram representing link).

<u>Proof</u> Need to check the behavior under the Reidemeister moves.

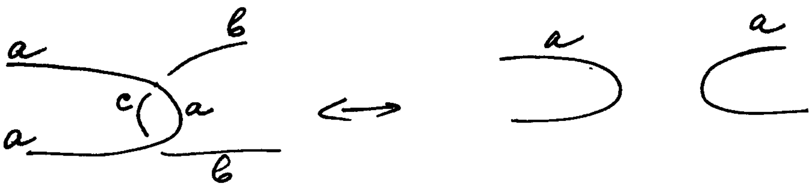
<u>R1:</u>



these ends have same color

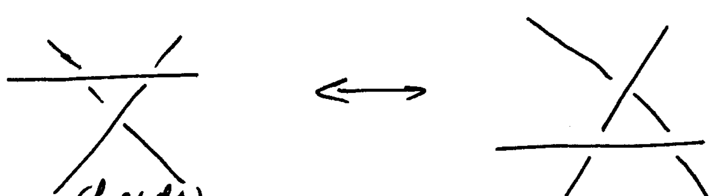
use this color here.

<u>R2:</u>



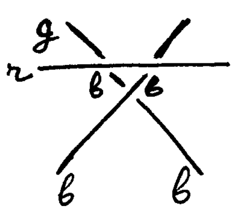
(if $a \neq b$, then $c \neq a \neq b$; if $a = b$, then $c = a = b$)

<u>R3</u>



1) all colors (of ends) are the same : use the same color on the r.h.s.

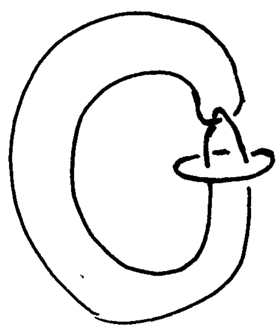
2) all colors (of ends) are different : use the same colours.

3) 

<u>Exercise:</u> Consider all cases. Complete the proof

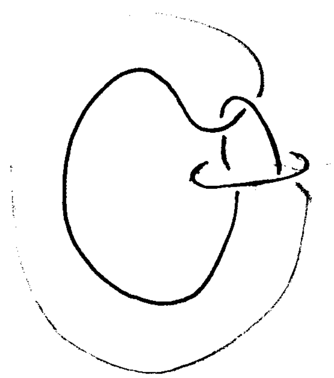
<u>Corollary.</u> Trefoil is <u>not</u> eq. to the unlink.

Ex. The Whitehead link:



2-component unlink:

$\tau = 9$.

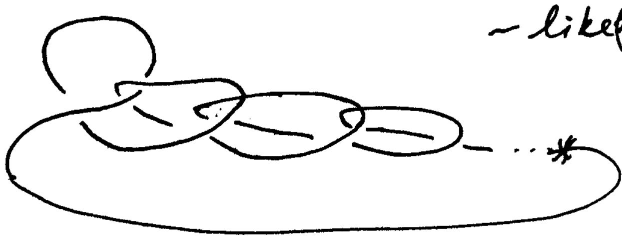


Thus, everything has to be of the same color!

$\Rightarrow$ τ distinguishes the Whitehead link from the 2-component unlink!

7/ Comments and discussion of lecture 1.

(1).



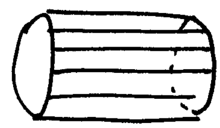
– like $\left(\sin \frac{1}{x}\right)$ (not an image of an injective map) $f: S^1 \to R^3$

– not a knot (at the limit point * the derivative $\frac{df}{d\theta}$ is <u>not</u> defined)

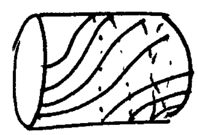
(2). <u>Ex.</u> Torus links:

$p > 0$, q – are integers

- form a cylinder with p strings along it:



- twist it through $\frac{q}{p}$ full twists:



- connect the cylinder to become a torus:

(3). Deformations of <u>knots</u>:

<u>Knots</u>: continuous $\boxed{\text{injective}}$ maps $f: S^1 \to R^3$

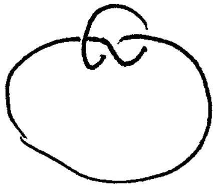
Two knots are equivalent if the corresponding ~~ku~~ maps are <u>homotopic</u>:

$$f_0: S^1 \to R^3, \quad f_1: S^1 \to R^3$$

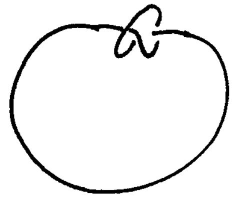
are homotopic if $\exists$ a cont. $\boxed{\text{injective}}$ map $F: S^1 \times I \to R^3$ s.t.

$$F(t,0) = f_0, \quad F(t,1) = f_1$$

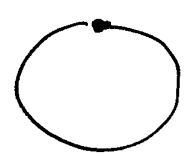
Not a good approach! All the knots are equivalent:



pull the string tighter



etc.



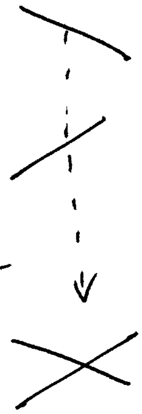
isotopy!

But we don't want to consider these two as equivalent!

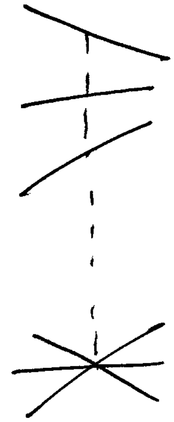
That's w[hy] we need ambient isotopy

8 (4). Projections of knots:

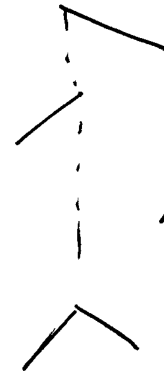
regular



irregular:



or



~~An~~ etc.

This can be resolved.

Turns out that this irregularities can always be resolved (perturbed knot and change the direction of the proj.)