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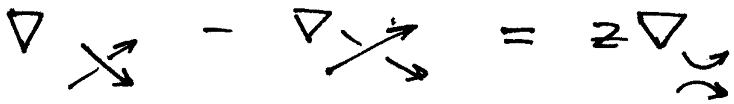
Lecture 9.

The Conway polynomial. (refined version of the Alexander poly)

Axioms: $\nabla_K(z)$ - polynomial in z.

① $\nabla_K(z)$ is an inv. of amb. iso. (for oriented links)

② $\nabla_{\bigcirc} = 1$

③ 

$$\nabla_{L^+} - \nabla_{L^-} = z\nabla_{L^0}$$

(L^+) (L^-) (L^0).

$\nabla_K(z) \in \mathbb{Z}[z]$ - the ring of polynomials in z with int. coeff.

Thus, $\nabla_K(z) = a_0(K) + a_1(K)\cdot z + \ldots + \cdots$,

where each $a_n(K)$ is an inv.

From axiom 3, it follow that

$$a_{n+1}(\cancel{K}) - a_{n+1}(\underset{L^-}{\cancel{K}}) = a_n(L).$$

(homework ex.)

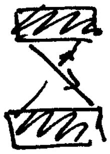
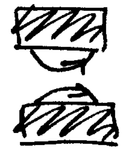
Assume that the axioms are consistent.

Ex. Split link (a link equivalent to the one with the diagram containing two nonempty parts that live in disjoint nblds).

E.g.:  $\Longleftrightarrow$ Whitehead $\bigcirc$.

Lemma. For a split link, $\nabla_L = 0$.

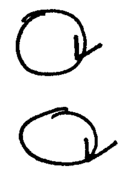
Proof.

 $\sim$   — split link

L^+ L^-

$\nabla_{L^+} = \nabla_{L^-}$. Thus, $\nabla_{L^0} = 0$.

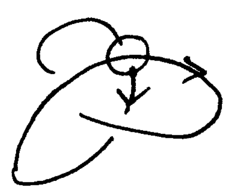
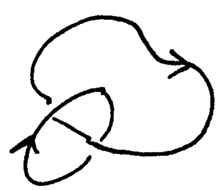
Ex.

$L^+ \sim L^-$ $\nabla_{L_0} = 0$.

More generally,
for an unlink with any number of components, ∇ is 0.
This is used in recursive calculations:

Ex. ①.

L^+ L^- L^0

$\nabla_{L^+} - \nabla_{L^-} = z\nabla_{L^0}$

$L^- \sim O \Rightarrow \nabla_{L^-} = 1$ $\Rightarrow \nabla_{L^+} = \nabla_{L^0} \cdot z + 1$.

$L^0 = L^0_+$ L^0_- W

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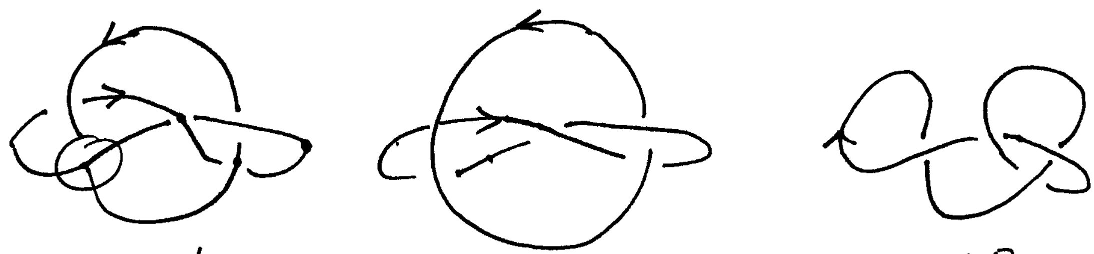
$$\nabla_{L_+^0} - \underbrace{\nabla_{L_-^0}}_{=\,1\ (\text{split})} = z\underbrace{\nabla_W}_{=\,1}. \quad \Rightarrow \quad \nabla_{L_+^0} = z.$$

Thus, $\boxed{\nabla_{L^+} = z^2+1.}$

(2).



L^+ $\qquad$ L^- $\qquad$ L^0

$lk(L^+)=0.$

$$\nabla_L - (-z) = z(1+z^2). \qquad \nabla_{L^-} = -z. \qquad \nabla_{L^0} = 1+z^2.$$

$$\Rightarrow \underline{\nabla_L = z^3.}$$

<u>Def</u> Let $c(L) = \begin{cases} 1 & \text{if } L \text{ has } 1 \text{ component} \\ 0 & \text{if } L \text{ has} \geq 2 \end{cases}$

<u>Thm.</u> $a_0(K) = c(K)$ for all links & knots.

$a_1(K) = \begin{cases} lk(K) & \text{if } K \text{ has } 2 \text{ components} \\ 0 & \text{otherwise.} \end{cases}$

Proof.

Apply axiom 3.

1. For a_0:

$$a_0(L^+) - a_0(L^-) = 0 \quad \Rightarrow \quad a_0 \text{ is inv. under strand switching.}$$

Since L^+ can be changed into unknot (or unlink), it follows that $a_0 = 1$ or 0.

$$\Rightarrow a_0(L) = C(L).$$

2. For a_1:

$$a_1(L^+) - a_1(L^-) = C(L) \quad \text{if } L^+ \text{ has 2 components.}$$

Exercise: Finish this proof.

~~3~~.

Ex. a_2 for the trefoil:

$$a_2(L^+) - a_2(L^-) = lk(L^\circ) = 1$$

$$a_2(L^-) = a_2(U) = 0, \qquad lk(L^\circ) = a_1(L^\circ)$$

$$a_2(L^+) = 1.$$

$a_2(K)$ computes self-linking number (obtained from links made by splicing crossings on K).