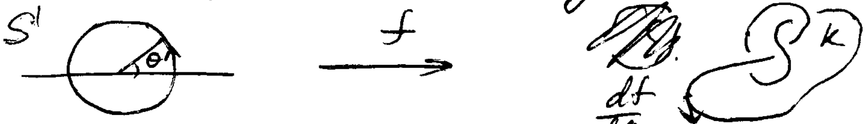
Lecture 1.

Knots, links; representing knots and links by diagrams;
crossings; universe of a knot;
unknot; orientation of knots and links;
(planar, ambient isotopy) Equivalence; Reidemeister moves and Reidemeister's theorem; Classification and notation for knots.
Invariants (number of components; linking number, examples)



Knot – A sub~~space~~set $K \subset \mathbb{R}^3$ is a <u>knot</u> if it is the image of a smooth injective map $f: S^1 \to \mathbb{R}^3$, with $\frac{df}{d\theta} \neq 0$. ($\frac{d^n f}{d\theta^n}$ exists for all n).

Ex. Simplest knot – the unknot:  (bounds a disc)

Link – a system of several knots: $L \subset \mathbb{R}^3$, $L = K_1 \cup \dots \cup K_n$, and $K_i \cap K_j = \emptyset$ for $i \neq j$.

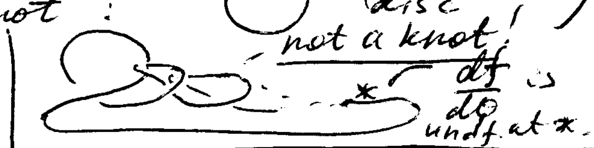
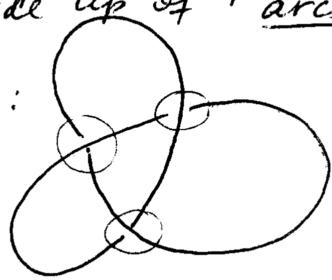
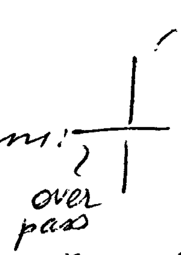
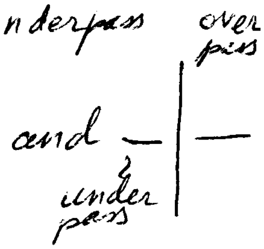


Diagram – planar picture of a knot (projection on a plane) made up of <u>arcs</u> and <u>crossings</u>

Ex. Trefoil:  3 crossings on this diagram.

Two types of crossings on a diagram: 

and 

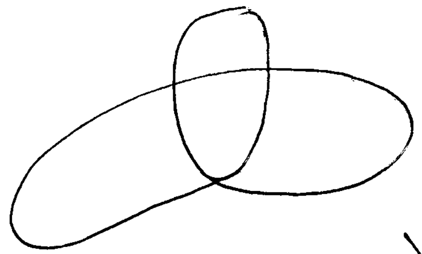
Not allowed: 

Such a diagram turns out to be sufficient to study properties of knots & links.

Knot $\Longleftrightarrow$ 4-valent planar graph together with choices of crossings.

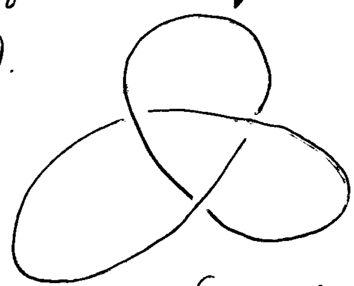
Universe – the graph of a knot without choices of crossings at vertices.

Ex.

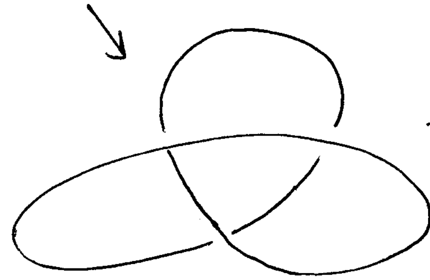


– Universe

n crossings in a universe in general, 2^n corresponding knots/links (many are equivalent and some are unknotted).



(Trefoil)



(Unknot)

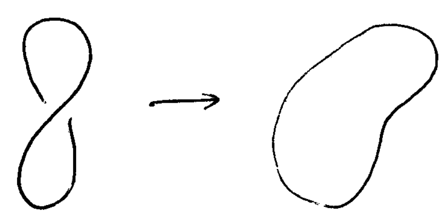
(Exercise Draw all possible knot diagrams coming from the trefoil universe (8 diagrams) – too simple)

Orientation – choice of direction on each component of a link.

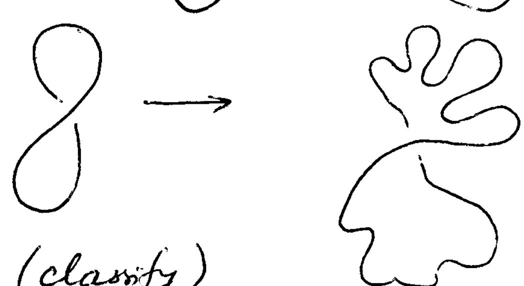
Ambient isotopy (Deformation of one knot into another in space)

Planar isotopy (Deformation of the diagram preserving the structure of a universe)

Ex. Ambient isotopy:



Planar isotopy:

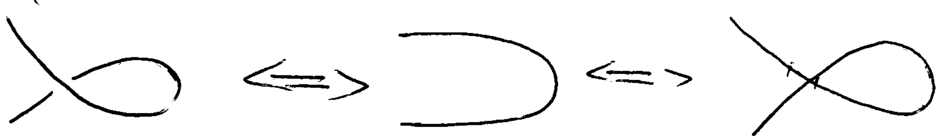


Goal: distinguish (classify) knots up to ambient isotopy.

Reidemeister moves.

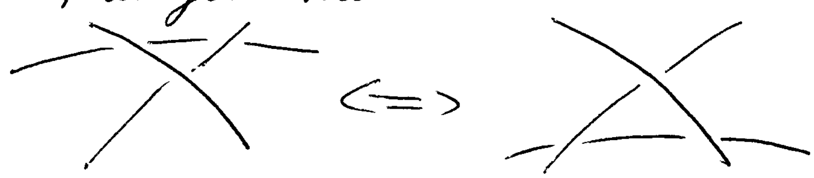
It turns out that 3 types of diagram moves are sufficient to describe ambient isotopy

I. Add (or remove) a curl:



II. Add (or remove) two consecutive under (over) crossings

III. Triangle move



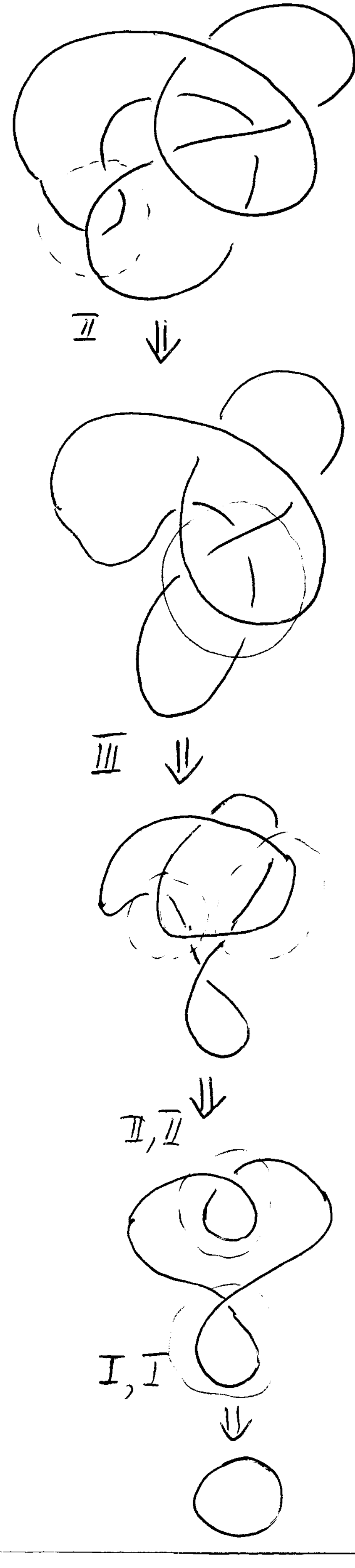
Reidemeister's theorem.

Two knots (links) in space can be deformed into each other by an ambient isotopy if and only if their diagrams can be transformed into each other by planar isotopy and Reidemeister moves.

Some consequences of Reidemeister moves:

Ex.

(Ambient isotopy via Reidemeister's moves)

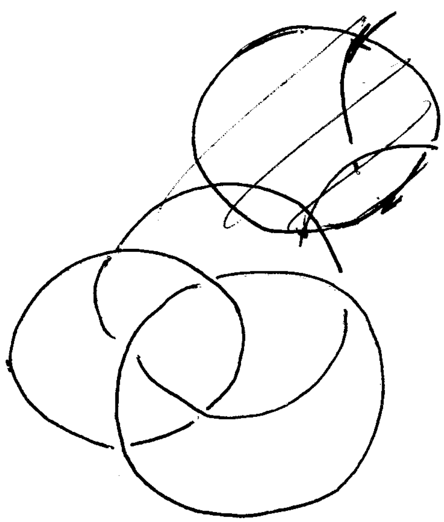


Number of components of a link.

The number of components remains the same under equivalence. $\Rightarrow$ Invariant.

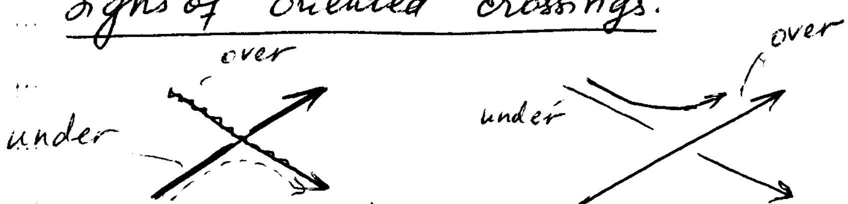
To determine the number of components:

- Choose a pt on an arc;
- walk along the diagram;
- each component is a completed cycle.



– 3 components.

Signs of oriented crossings:



(crossing signs)

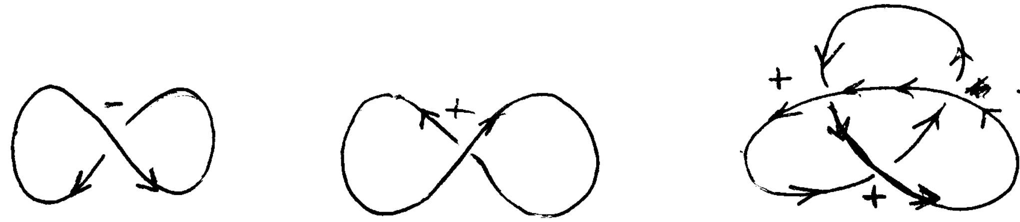
(~~left to~~ right hand) $\varepsilon = +1$

$\varepsilon = -1$ (left hand)

Writhe of (a diagram of) a knot – sum of the crossing signs

$$w(K) = \sum_{\substack{\text{crossings} \\ (c_1, \dots, c_n)}} \varepsilon(c_i)$$

Ex.



$w = -1$ $\qquad$ $w = 1$ $\qquad$ $w = 1$

Q: Is writhe an invariant of a knot?
(To answer, we need to look at how the writhe changes under the Reidemeister moves:

R1:  – not preserved.

$w = -1$ $\qquad$ $w = 0$

R2: 

– preserved

$w = 0$ $\qquad$ $w = 0$

R3: 

– preserved.

$w = 1$ $\qquad$ $w = 1$

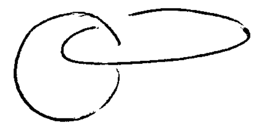
It follows that the writhe is preserved only under R2 and R3.

The linking number.

Consider a link with 2 components, α and β.
Let $\alpha \sqcap \beta$ be the set of crossings of α with β.
(not including self-crossings)

$$lk(\alpha, \beta) \doteq \frac{1}{2} \sum_{c \in \alpha \sqcap \beta} \operatorname{sign}(c)$$

(one-half of the sum of crossing signs of one curve with another).

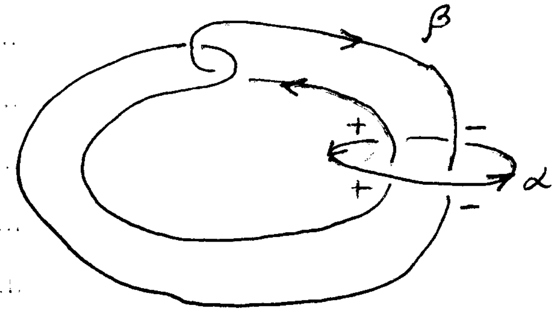
Thus  is linked

Examples.

$lk(\alpha,\beta) = -1$ $\qquad$ $lk(\alpha,\beta) = 1$

HW: Check that the linking number is invariant under Reidemeister moves and is, therefore, an invariant

Example. The Whitehead link:



$$lk(\alpha,\beta) = \frac{1}{2}(1+1-1-1) = 0$$

– The linking number is 0, but, it is clearly linked!