The Enhanced Linking Number

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March 21, 2005

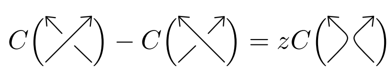
Abstract

The simplest of questions that arise in the study of links turn out to be the most difficult. Perhaps the most immediate of these questions is whether there exists a computable method for determining equivalence among links. After all, it seems intuitive that such a method should exist. While no such method is currently known, we can at least, in some cases, conclude that two links are inequivalent. In this paper we will explore some properties of the enhanced linking number and its relation to the standard linking number, and how they relate to this fundamental question.

1 The Conway Polynomial

In this section we see how the Conway polynomial is used to construct the enhanced linking number function, from which we derive a defining relation.

The Conway polynomial $C(z)$ is a polynomial in one variable (conventionally z). It is succinctly defined by the following relations (the first relation is called a *skein relation*):

 (1)

$$C\left(\bigcirc\right) = 1 \tag{2}$$

One may associate to each link (or a knot, which is really just a special case of a link), such a polynomial $C(z)$. By Conway's work, any two equivalent (*ambient isotopic*) links have the same associated Conway polynomial.

In particular, we see that each coefficient in the Conway polynomial is also invariant. We may therefore define another invariant as follows.

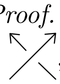
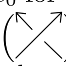
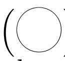
Let $L = (K, J)$ be a link of two components, K and J, and let $C(L) = c_0 + c_1 z + c_2 z^2 + \ldots$ be the Conway polynomial of L. We define λ to be the real valued function given by the relation (where $c_1(L)$ denotes the c_1 coefficient of $C(L)$, and so on)

$$\lambda(L) = c_3(L) - c_1(L)(c_2(K) + c_2(J)). \tag{3}$$

We know that λ is a real-valued function since (2) is real (Note that (1) is used to reduce any link to terms involving (2)).

It turns out that λ may also be defined in terms of a skein relation. In order to derive this relation we will need to know some simple results regarding the Conway polynomial.

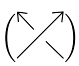
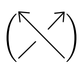
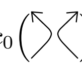
Theorem 1.1. For a knot K, $c_0(K) = 1$.

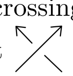
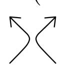
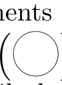
Proof. If K is the unknot, then (2) gives us the result trivially. Otherwise, $K =$ , (we picture only a part of K relevant to the argument, but keep in mind that this really represents an entire knot). Note that c_0 for the right-hand side of (1) is zero (because of the z). So we have that $c_0\Big($  $\Big) = c_0\Big($  $\Big)$. This tells us that c_0 for a knot is invariant under crossing changes. Using the fact that a knot may transformed into the unknot through crossing changes, and $c_0\Big($  $\Big) = 1$, it follows that $c_0(K) = 1$.

We denote the linking number function by lk. It's simply defined as half of the sum of the signs of all non-self crossings.

Theorem 1.2. For a two-component link L, we have $c_1(L) = lk(L)$.

Proof. (1) implies the relation

$c_1\Big($  $\Big) - c_1\Big($  $\Big) = c_0\Big($  $\Big)$ (4)

If the crossing pictured in the leftmost link in (4) is not a self-crossing, then the link pictured on the right side becomes a one-component link, and by Theorem 1.1, the right side of (4) is 1. If the crossing is a self-crossing, then the right side represents a three-component link (remember that  represents a two component link, so splicing to make  adds one component). Looking at (1) for a three-component link, we note that it takes at least two splicings to obtain the first nonzero component (since $C(z)$ for any unlink with two or more components is zero - see Theorem 2.1), which is the unknot. Since in each splicing we multiplied by z, the end result will have lowest power z^2, so that $c_0 = c_1 = 0$. We conclude that the right side of the equation in (4) is 0. We see that this is exactly the behavior of the linking number. (A self-crossing change does not change the value, and a crossing change between two different components changes the value of the linking number by ± 1). Using the fact that $c_0\Big($  $\Big) = 0$ we obtain the result.

Similarly we may prove the following

Theorem 1.3. For a three-component link $L = (A, B, C)$ we have

$$c_2(L) = lk(A,B)lk(A,C) + lk(B,A)lk(B,C) + lk(C,A)lk(C,B)$$

The proof of this theorem is lengthy and involves considering many more cases than was done in Theorem 1.2, so it is not given here.

Theorem 1.4. Let $L = (K, J) = \left(\quad , J\right)$, $L' = (K', J) = \left(\quad , J\right)$ be two links of two components. Then the function λ defined above satisfies

$$\lambda\left(\quad , J\right) - \lambda\left(\quad , J\right) = lk\left(\quad , J\right) lk\left(\quad , J\right). \tag{5}$$

It is crucial that the crossing changes are self-crossing changes, which is clearly indicated by the notation.

Proof. Clearly $lk(L) = lk(L') = lk\left(\quad , J\right) + lk\left(\quad , J\right)$. Let $lk(L) = n$. Then

$$\lambda\left(\quad , J\right) - \lambda\left(\quad , J\right) = \tag{6}$$

$$c_3(L) - c_1(L)(c_2(K) + c_2(J)) - \Big[c_3(L') - c_1(L')(c_2(K') + c_2(J))\Big] = \tag{7}$$

$$c_3(L) - n(c_2(K) + c_2(J)) - \Big[c_3(L') - n(c_2(K') + c_2(J))\Big] = \tag{8}$$

$$c_3\left(\quad , J\right) - c_3\left(\quad , J\right) - n(c_2\left(\quad \right) - c_2\left(\quad \right)) = \tag{9}$$

$$c_2\left(\quad , J\right) - n(c_1\left(\quad \right)) = \tag{10}$$

$$c_2\left(\quad , J\right) - (lk(\quad , J) + lk(\quad , J))(lk(\quad)) = \tag{11}$$

$$lk(\quad , J)lk(\quad , J) \tag{12}$$

Application of (3) results in (7). Cancellation of $nc_2(J)$ gives (10). By the principle in (4) we obtain (11). We get (12) because for a three-component link $L = (A, B, C)$ we have $c_2(L) = lk(A,B)lk(A,C) + lk(B,A)lk(B,C) + lk(C,A)lk(C,B)$. The proof of this is similar to Theorem 1.2 but but involves considering many more cases, so it will not be proved in detail.

Using this skein relation (5) we can compute the value of λ for any two-component link, as we shall see in Section 4. We will still need, however, the values of λ on two simple links.

2 The Unlink and the Hopf Link

It is instructive to compute the values of λ on some simple links, which will be needed for the proofs in Section 3.

Theorem 2.1. For any unlinked two-component link L, $\lambda(L) = 0$.

Note: By unlinked we mean that the diagram consists of two disjoint diagrams.

Proof. Any two unlinked knots K_1, K_2 can be viewed as the smoothed component in (1), where the two arcs are from separate components. Then joining the two components, we obtain two knots where the one can be transformed into the other through two "large" Reidemeister type-1 moves as pictured in Figure 1. The circled K_1, K_2 represent arbitrary knots, and the point on K_2 is used to indicate how K_2 is rotated.

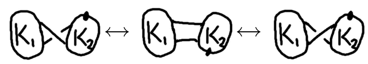


Figure 1: The "Large" Reidemister 1 move.

So $C(K_1) - C(K_2) = 0 = zC(L)$. So in particular $c_1(L) = c_3(L) = 0$ and from (3), we conclude that $\lambda(L) = 0$.

Theorem 2.2. For the Hopf links H^+, H^- with linking numbers 1 and -1, respectively, $\lambda = 0$.

Proof. Using (1) and (2) we compute $C(H^+) = z, C(H^-) = -z$. Application of (3) gives the result.

3 The Standard Form

The links pictured in Figure 2 have linking number 5. We denote them by T_5^+ and $T_5'^+$. (Also let T_5^-, $T_5'^-$, denote the same links but where the orientation on the unknot is switched so that the linking number becomes -5, see Figure 4). It turns out that any two component link with linking number 5 can be put into either form in Figure 2, and any two component link with linking number -5 into a form in Figure 4. Let T_n^+ and $T_n'^+$ be links like those in Figure 2 (where $n = 5$), in which one component is the unlink and the other component wraps monotonically around the first, such that this component produces n local maxima. The following generalization then holds:

Theorem 3.1. An oriented two-component link with linking number n ($n = 1, 2, \ldots$) can be transformed by self-crossing changes into T_n^+ (and also

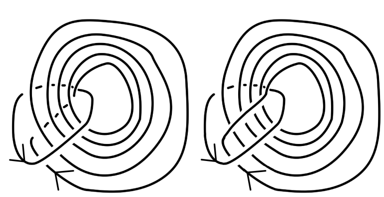


Figure 2: The standard links T_5^+ and $T_5'^+$ (respectively).

into T'^{+}_n).

Note that the Hopf link results when $n = 1$, the Whitehead for $n = 2$, and that T_0 is not defined.

Because the proof is long and technical, a complete proof will not be presented. However the idea behind the proof is simple enough. First we need to show that one component can be transformed by self-crossing changes into the unknot *without changing any crossings between the two components.* Then need to show that the other may be wound monotonically around the first.

Theorem 3.1 has an obvious analog involving T^- and T'^-. The statement of this theorem is left to the reader as a simple exercise.

It follows that for any two-component links with the same linking number, one of them may be transformed into the other through self-crossing changes. (First transform both to the standard form, and then do the reverse transformation but on different components.) The following corollary is not pertinent to our current discussion but because it simplifies the calculation of the standard linking number lk it is nonetheless included.

Corollary. The linking number for a two component link may be computed by counting only the signs on which one component crosses *over* the other component. This is clear because it is true for the link in the standard form, into which any link may be transformed in some number of steps that preserve the linking number (self-crossing changes do not affect the value of the linking number).

Theorem 3.2. For all nonzero integers n, m, we have $\lambda(T^+_n) = \lambda(T^+_m)$.

Proof. The main idea of this proof is that splicing at the crossing pictured in Figure 3 will result in the unknot, and the crossing change here results in T^+_{n-1}. The details are as follows.

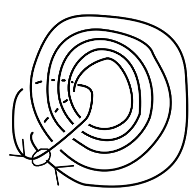


Figure 3: T^+_5 where a crossing is marked.

Because each of the components in standard form is the unknot, a quick glance at (2) shows that the c_2 will be zero. Thus λ (as computed in (3)) for any link in the standard form is completely determined by c_3. Smoothing using (1) at one of the crossings will result in the unknot (for which another glance at (2) shows that c_2 is zero), so that the third Conway coefficient is equal for the link and the link whose linking number is one less. So for all n, m, $T^+_n = T^+_m$.

Theorem 3.3. For all nonzero integers n, m, we have $\lambda(T'^-_n) = \lambda(T'^-_m)$.

Proof. The argument is identical with that in Theorem 3.2. Here T' becomes the unknot after splicing, and not T, as in Theorem 3.2.

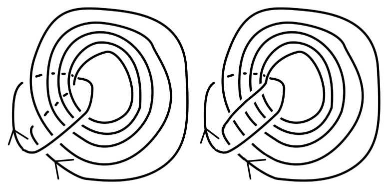


Figure 4: The standard links T_5^- and $T_5'^-$ (respectively).

4 An Algorithm for λ

The recursive property (5) of λ can be used to calculate the value of λ for all two-component links (except for those outlined in Section 2) without explicit reference to the definition in (3). To do so we need the following result, which follows immediately from Theorems 3.2 and 3.3.

Theorem 4.1. For all n, $\lambda(T_n^+) = 0$ and $\lambda(T_n'^-) = 0$.

Proof. Both T_1^+ and $T_1'^-$ are Hopf links, for which by Theorem 2.2, we have $\lambda = 0$.

Theorem 4.2. We can use the skein relation of λ to compute λ for any two-component oriented link.

Proof. We give an algorithm for computing λ. Let $L = (K_0, J_0)$ be our link. Assume $lk(L) = n > 0$ (the case $n < 0$ proceeds similarly). By Theorem 3.1 there exists some sequence of k self-crossing changes transforming K into the unknot. Write the states in this transformation of L as $\{(K_0, J_0), (K_1, J_0), (K_2, J_0), \ldots, (K_k, J_0)\}$. Then, there is a sequence (denoted $\{(K_k, J_0), (K_k, J_1), (K_k, J_2), \ldots, (K_k, J_j)\}$) of j self-crossing changes on J so that J wraps monotonically (as in Theorem 3.1) around K_k. Then the skein relation (5) tells us that

$$\lambda(K_i, J_0) = \lambda(K_{i+1}, J_0) \pm a_i \tag{13}$$
$$\lambda(K_k, J_i) = \lambda(K_k, J_{i+1}) \pm b_i, \tag{14}$$

where

$$a_i = \pm lk\Big(\nwarrow\!\rangle, J\Big)\, lk\Big(\langle\!\nearrow, J\Big) \tag{15}$$

and

$$b_i = \pm lk\Big(K_k \Big) lk\Big(K_k, \Big). \tag{16}$$

The segments in (15) and (16) are the splicings at the i^{th} and j^{th} crossings where crossing changes were made. Clearly if the i^{th} crossing change results in a positive crossing, we obtain a negative sign (and otherwise a positive sign). We then obtain the following computations

$$\lambda(K_0, J_0) = \lambda(K_k, J_0) + \sum_{i=0}^{k-1} \pm a_i, \tag{17}$$

$$\lambda(K_k, J_0) = \lambda(K_k, J_j) + \sum_{i=0}^{j-1} \pm b_i \tag{18}$$

so that finally

$$\lambda(K_0, J_0) = \sum_{i=0}^{k-1} \pm a_i + \sum_{i=0}^{j-1} \pm b_i \tag{19}$$

since (K_k, J_j) is T_n^+, for which $\lambda = 0$ (Theorem 3.2). The a_i and b_i are real so we have the result.

Having shown that (5) may be used to calculate λ, we have the following

Theorem 4.3. Let λ' be the real valued function defined by the skein relation in (5), and such that $\lambda'(H^+) = \lambda'(H^-) = \lambda'\Big(\bigcirc\Big) = 0$. (where H^+ and H^- are the standard oriented Hopf links). Then λ' is equivalent to the λ defined in (3).

This is clear from Theorems 4.2 and 2.2.

5 Examples

Consider the Whitehead link in Figure 5:

This is in fact T_2^-. We apply Theroem 4.2, noting that it takes only one crossing change to obtain $T_2'^-$.

$$\lambda(T_2^-) = \lambda(T_2'^-) - lk(T_1^-) lk(T_1^-) = (-1)(-1) = 1 \tag{20}$$

Theorem 4.2 requires that we find a sequence of self-crossing changes on each component to bring the link into a standard form. While we are guranteed that such a sequence exists, it is still unknown if there exists a procedure to discover it. We shall explore this issue in some detail.

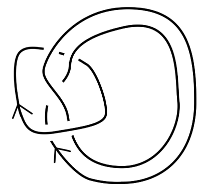


Figure 5: The Whitehead link.

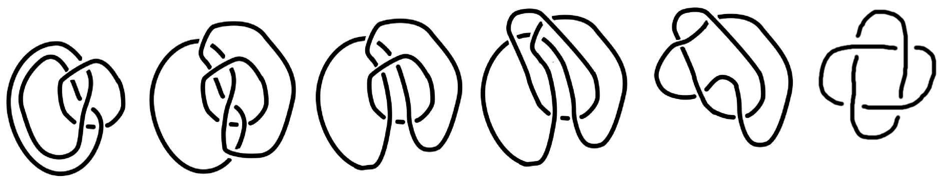


Figure 6: Reversing the roles of components.

A consequence of Theorem 3.1 is that it is always possible to switch the role of the component that winds with the one that is the unknot.

This example shows that the role of the components may be switched. Since the rightmost diagram is symmetric with respect to a 90 degree rotation, performing this rotation followed by execution of the moves going in the leftward direction results in an identical standard link, but where the role of the two components is reversed.

Although this is a simple example, it follows from Theroem 3.1 that a role-reversal may be performed on any link. In practice, however, it may be very difficult to do this.

In some cases it makes no difference as to which component is chosen to deform into the unknot. The link in Figure 7 is symmetric; the role of the two components can be interchanged by a 180 degree rotation. Yet to bring this link into the standard form proves quite difficult. Here the recursive relation (5) for computing λ may not be of much help.

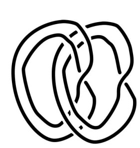


Figure 7: A link for which computing λ as outlined in Theorem 4.2 is difficult.

6 Special Links, Indistinguishables

We now return to the original question: how can λ be used to distinguish links (in particular, a given link from the unlink)?

Figure 8: Another Whitehead link.

The link in Figure 8 is a nontrivial link with linking number zero. It is readily seen that one self-crossing change creates the unlink. So λ for the pictured link is simply $-lk(H^+)lk(H^-) = 1$. So λ is quite useful for the simplest examples of nontrivial links with linking number zero.

Recall that λ for the other Whitehead link in Figure 5 had the same value.

Because the Conway polynomial fails to distinguish all links, λ fails to distinguish links. Figure 9 below gives an example of two inequivalent links with the same Conway polynomial and the same value of λ

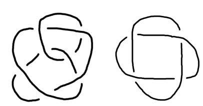


Figure 9: These two prime links (7^2_7 and 4^2_1 respectively), have the same Conway polynomial $2z + z^3$.

However, as we have seen, for the simplest link, the Hopf link, $\lambda = 0$. So where the linking number fails to distinguish some simple links from the unlink, λ has the same failings.

There are also links for which both lk and λ are zero. The first nontrivial prime link for which this is true is 8^2_{10}.

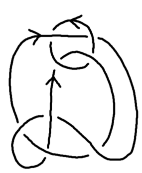


Figure 10: 8^2_{10}, for which λ and lk are zero.

7 Going Further

How some questions that arise might be answered.

There are many practical problems in computing λ for a link. While it may always be computed through the Conway polynomial, doing that is not an easy task. A procedure for choosing the component to be transformed into the unknot and for choosing the subsequent self-crossings to have the other component monotonically winding around it would be provide a powerful tool for investigating linking. Also, since λ depends on c_1, c_2, c_3, such a method would also shed light on the behavior of the Conway polynomial.

References

[1] Charles Livingston, Enhanced Linking Numbers, *American Mathematical Monthly*, May 2003, pp. 361-385.

[2] Helmut Doll and Jim Hoste, A Tabulation of Oriented Links. With microfiche supplement, *Mathematics of Computation*, Oct 1991, pp. 747-761.