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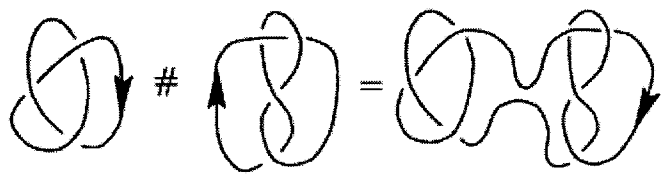
Prime Factorization of Knots

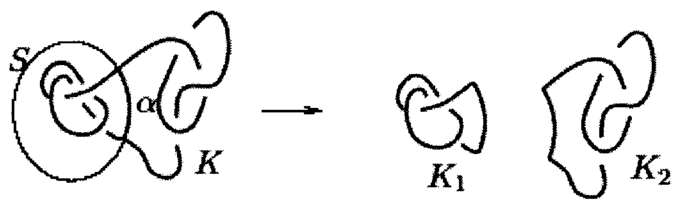
Abstract

A knot is called *prime* if it can not be represented as a connected sum of two knots such that both of these are knotted. Using the notion of a *Seifert surface* of a knot, we define a knot's *genus,* an additive invariant which allows to prove the existence of prime knots. Then, after defining an equivalence relation on all possible ways of factoring a knot, we will show that there is only one equivalence class. Hence we show that every knot has a unique (up to order) factorization into prime components.

Definitions: *Connected Sum*:



Factoring a knot:



A knot is *prime* if it cannot be factored into two knotted components.

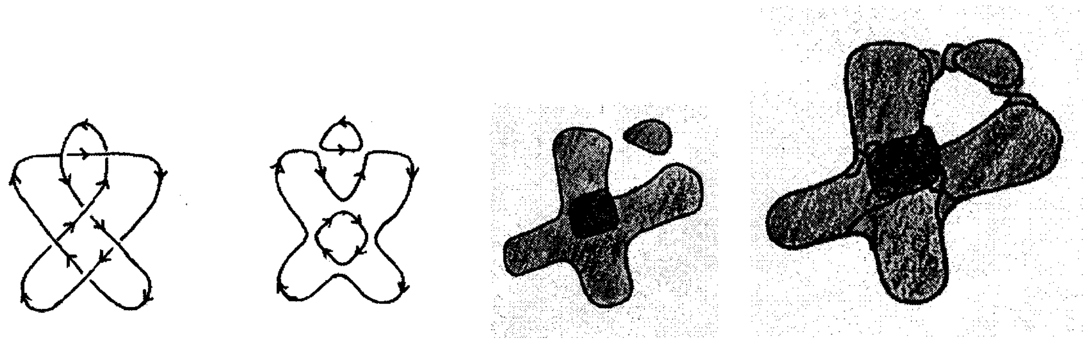
Existence of prime knots:

Definition: A Seifert surface S_K of a knot K is an orientable surface (in S^3) that has boundary K.

Seifert's Algorithm:

1) Given a projection of a knot, orient it, and resolve all crossings.
2) Now consider each component as the boundary of a disk at a different level in 3-space
3) Finally, reconnect the disks where the crossings were with a strip, giving it a half-twist.

Example:



Orientability:

Note that we can two-color across any bridge, whether the circles are nested or not.

	Top	Bottom
If the Seifert circles run clockwise:	dotted	lined
If the Seifert circles run counterclockwise:	dotted	lined

Definitions: The *associated compact surface* $\hat{S}_K$ of a surface S_K (orientable) is obtained by "sewing" a disk along each boundary component. (Note that it is unique up to homotopy).

The *genus* of a knot K is the minimal genus over all associated compact surfaces of K.

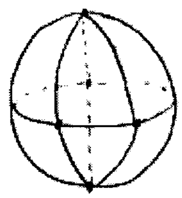
In order to compute the genus of a surface, we use a surface invariant (under homotopy) called the *Euler Characteristic* of a surface S, denoted $\chi(S)$.

First, triangulate the surface S. Then count the number of V̲ertices, E̲dges and F̲aces.

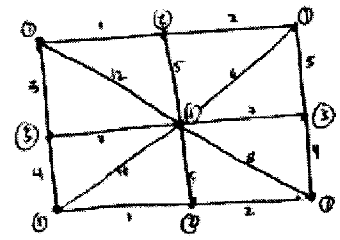
Then

$$\chi(S) = V - E + F \qquad (1)$$

Examples:



$\chi(\mathbf{S}^2) = 5 - 9 + 6 = 2$



$\chi(\mathbf{T}^2) = 4 - 12 + 8$

Now notice that if we take the connected sum of two triangulated surfaces, we get the relation:

$$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2 \qquad (2)$$

And by induction on the genus g of a surface:

$$\chi(S) = 2 - 2g \Leftrightarrow g(S) = 1 - \chi(S)/2 \qquad (3)$$

Now, to avoid having to triangulate compact surfaces, notice that:

$$\chi(\hat{S}_K) = \chi(S_K) + 1 \qquad (4)$$

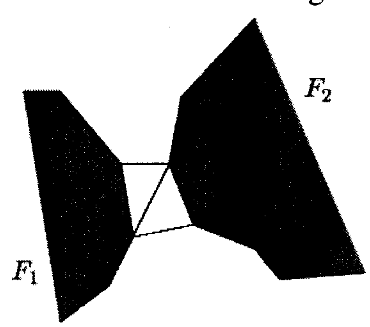
and then using (3)

$$g(\hat{S}_K) = 1 - \frac{\chi(S_K)+1}{2} \qquad (5)$$

Proposition: Let s be the number of Seifert circles and c be the number of crossings of a projection of a knot K. Then:

$$\chi(S_K) = s - c \qquad (6)$$

Choose a triangulation so that the vertices are at the edge of each "strip".



Then

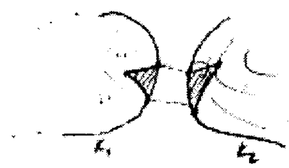
$$\chi(S_K) = 1{*}s + c(0 - 3 + 2) = s - c$$

and by (5)

$$g(\hat{S}_K) = 1 - \frac{s - c + 1}{2} \qquad (7)$$

Theorem: $g(K_1\# K_2) = g(K_1) + g(K_2)$ (8)

First construct a minimal Seifert surface $\hat{S}_{K1}$ and $\hat{S}_{K2}$ for the knots K_1 and K_2. Triangulate them in such a way that at least two vertices appear "close enough" along each knot. Then take the direct sum of the surfaces using a triangle that includes the chosen vertices.



Then, by (2) $\chi(\hat{S}_{K1}\# \hat{S}_{K2}) = \chi(\hat{S}_{K1}) + \chi(\hat{S}_{K2}) - 2$

Using (3)

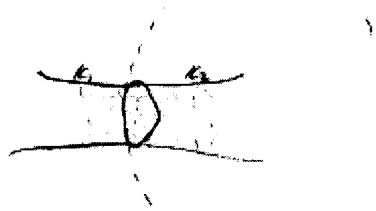
$$g(\hat{S}_{K_1}\#\hat{S}_{K_2}) = 1 - \frac{\chi(\hat{S}_{K_1}) + \chi(\hat{S}_{K_2}) - 2}{2}$$
$$= 1 - \frac{\chi(\hat{S}_{K_1})}{2} + 1 - \frac{\chi(\hat{S}_{K_2})}{2}$$
$$= g(\hat{S}_{K_1}) + g(\hat{S}_{K_2})$$

Since $\hat{S}_{K1}\#\hat{S}_{K2}$ is a Seifert surface for $K_1\# K_2$,

$$g(K_1\# K_2) \leq g(K_1) + g(K_2)$$

Now build a Seifert surface $\hat{S}_{(K1\# K2)}$ of minimal genus for $K_1\# K_2$. Then select a sphere that will separate the knot into its factors. It can be selected so that all of its intersections with $\hat{S}_{(K1\# K2)}$ are 1-dimensional. Triangulate the surface and the interiors of the intersections between the surface and the dissecting sphere. Then split the surface into $\hat{S}_1$ and $\hat{S}_2$, "capping off" all holes with the interiors of the intersections.

Let's examine the case when the intersection with the sphere is a circle. All other cases are similar.



$$\chi(\hat{S}_1) + \chi(\hat{S}_2) = (V_{(S1\# S2)} + n + 2) - (E_{(S1\# S2)} + 3n) + (F_{(S1\# S2)} + 2n)$$
$$= \chi(\hat{S}_{(K1\#K2)}) + 2$$

Using (3):

$$g(\hat{S}_1) + g(\hat{S}_2) = 1 - \frac{\chi(\hat{S}_1)}{2} + 1 - \frac{\chi(\hat{S}_2)}{2}$$
$$= 2 - \frac{\chi(\hat{S}_1) + \chi(\hat{S}_2)}{2}$$
$$= 2 - \frac{\chi(\hat{S}_{(K1\# K2)}) + 2}{2}$$
$$= 1 - \frac{\chi(\hat{S}_{(K1\# K2)})}{2} = g(S_1 \# S_2)$$

hence $$g(K_1) + g(K_2) \leq g(K_1 \# K_2)$$

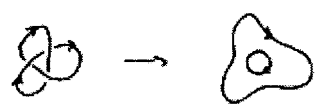
and the theorem is proven.

Note that a knot has genus 1 if and only if it is the unknot:

For the direct statement $g = 1 - (1-0+1)/2 = 0$ (by (7))
For the indirect statement, $0 = 1-(\chi(S_K) + 1)/2$ (by (5))
$\Rightarrow \chi(S_K) = 1 \Rightarrow S_K$ is a disc.

We conclude that **knots do not have inverses** under the connected sum operation.

Let's compute the genus of the trefoil:

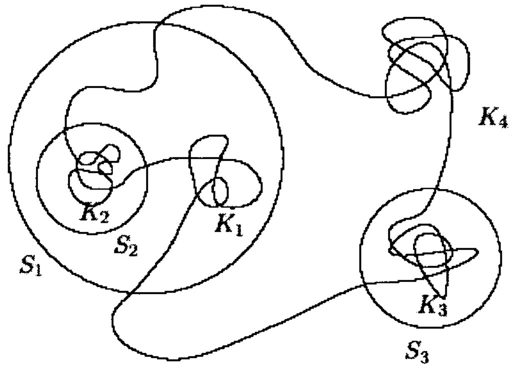


By (7) $g(K) = 1 - (2-3+1)/2 = 1$

We conclude that the trefoil is prime.

Uniqueness

Definition: A *dissecting sphere system* **S** of a knot K is a collection of spheres that do not intersect and that "assign one prime factor per region"



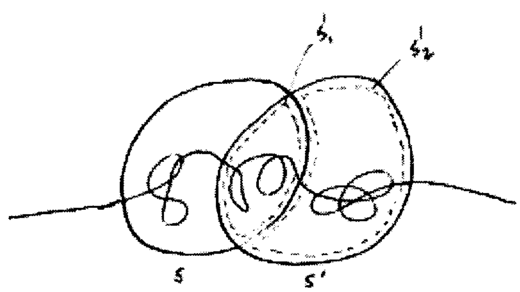
We say that **S** ~ **S'** if **S** and **S'** determine the same factorization of K.

We show that for any **S** and **S'** that dissect a knot K **S** ~ **S'**, by doing induction on the number n of components of **S**∩**S'**.

If n = 0: Notice that if **S**∩**S'** = Ø, **S**~**S'**, since we can interchange some of the spheres to turn **S** into **S'**.

If n > 0: Assume the result is true for (n – 1) components. Reduce the number of components as follows:

Choose a pair of intersecting spheres s and s' in **S** and **S'** respectively. Consider the spheres s ∩ s' and s'\ s. They cannot both contain knotted components since the knot in s' is prime. They cannot both determine unknots since otherwise the resulting knot would be the unknot. Replace s' with ŝ' and shrink it slightly. We now have reduced the number of intersections by one, and by induction, the proof is complete.



Conclusion:

Be aware that this theory does not distinguish knots that have different orientations. This theory is in line with that that was used to classify all knots

Also, the notion of Seifert surface and genus is constructed in exactly the same way for links as we have done for knots. The same goes for prime factorization of links.

The existence part of the proof can be shortened considerably in the following way:

> All torus knots are equivalent to the standard torus knots that we saw in homework 1. Draw the square with opposite sides identified in the proper way, and embed the knot in the torus. Think about all of the ways that a dissecting sphere can intersect the torus. There should not be any way to capture knotted component.

An interesting proof that is not included here shows that when one applies Seifert's algorithm to alternating projections of knots, the resulting Seifert surface is of minimal genus.