Fundamental Groups and Knots

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Abstract: In this paper, I will give an introduction to fundamental groups in a topological space, and the application of it as a knot invariant, called the knot group.

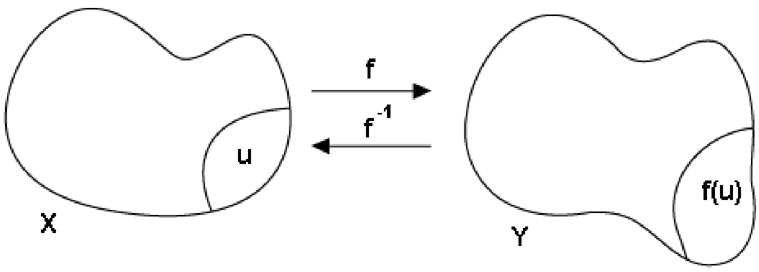
Introduction to Fundamental Groups

Definition: Let X and Y be topological spaces. Let $f: X \to Y$ be a bijection. If both the function f and the inverse function:

$$f^{-1}: Y \to X$$

are continuous, then f is called a *homeomorphism*.

The condition that f be continuous says that for each open set U of Y, the inverse image of U under the map f is open in X.



Definition: If f and f' are continuous maps from a topological space X into a topological space Y, we say that f is *homotopic* to f' if there is a continuous map $F: X \times [0,1] \to Y$ such that

$$F(x,0) = f(x) \text{ and } F(x,1) = f'(x) \qquad \text{for each } x.$$

The map F is called a *homotopy* between f and f'.

Moreover, we can consider a homotopy as a continuous deformation of the map f to the map f', as t represent the time from 0 to 1.

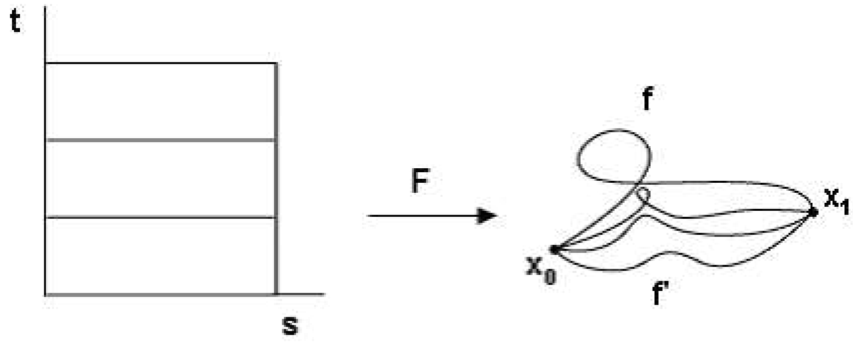
In particular, if we have X = [0,1] in the above definition, and same end points x_0 and x_1 for f and f', we obtain:

Definition: Two paths f and f' with fixed end points x_0 and x_1 mapping the interval [0,1] into X, are said to be *path-homotopic*, and if there is a continuous map $F: [0,1] \times [0,1] \to X$ such that

$$F(s,0) = f(s) \quad \text{and} \quad F(s,1) = f'(s),$$
$$F(0,t) = x_0 \quad \text{and} \quad F(1,t) = x_1$$

for each s in [0,1] and each t in [0,1]. We call F a *path-homotopy* between f and f'.

Path-Homotopy between f and f':



For this paper, we will only be looking at path-homotopy, so we will simply call it homotopy for convenience.

Definition: If f is a path in X from x_0 to x_1, and if g is a path in X from x_1 to x_2, we define the *product* $f \bullet g$ of f and g to be the path h given by the following equation:

$$h(s) = \begin{cases} f(2s) & \text{for } s \text{ in } [0,0.5], \\ g(2s-1) & \text{for } s \text{ in } [0.5,1]. \end{cases}$$

Geometrically, $f \bullet g$ is the concatenation of paths f and g.

The product operation on paths induces a well-defined operation on homotopy classes, defined by:

$$[f] \bullet [g] = [f \bullet g].$$

Definition: Let X be a topological space. Let x_0 be a point of X. A path in X that begins and ends at x_0 is called a *loop* based at x_0. The set of homotopy classes of loops based at x_0, with the operation $\bullet$, is called the *fundamental group* of X relative to the *base point* x_0. It is denoted by $\pi_1(X,x_0)$.

Definition: A topological space X is path-connected if for any two points in X we can find a path which connects them and entirely lies in X.

For path connected topological space we have the following theorem:

Theorem: If X is path-connected and x_0 and x_1 are two points of X, then $\pi_1(X,x_0)$ is isomorphic to $\pi_1(X,x_1)$.

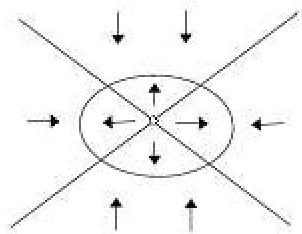
Thus, it makes sense to talk about the fundamental group $\pi_1(X)$ of a path-connected topological space X without a reference to the base point.

Definition: Let A be a subspace of X. We say that A is a *deformation retract* of X if the identity map of X is homotopic to a map that carries all of X into A, in such a way that each point of A remains fixed during the homotopy. This means that there is a continuous map H: X x [0,1] → X such that

(1) $H(x,0) = x$	for all x in X,
(2) $H(x,1)$ is in A	for all x in X,
(3) $H(a,t) = a$	for all a in A.

The homotopy H is called a *deformation retract* of X onto A. The map r: X → A defined by the equation $r(x) = H(x, 1)$ is a *retraction* of X onto A, and H is a homotopy between the identity map of x and the map j o r, where j: A -> X is the inclusion.

Example: Consider $R^2\setminus(0,0)$. We assert that this space has the unit circle as a deformation retract. The retraction is shown in the picture below:



Definition: Let h: $(X,x_0) \to (Y,y_0)$ be a continuous map. Define

$$h_*: \pi_1(X,x_0) \to \pi_1(Y,y_0)$$

by the equation

$$h_*([f]) = [h \circ f].$$

The map h_* is called the *homomorphism* induced by h, relative to the base point x_0.

Lemma: Let $f_0, f_1: (X,x_0) \to (Y,y_0)$ be continuous maps. If f_0 and f_1 are homotopic, and if the image of the base point x_0 of X remains fixed at y_0 during the homotopy, then $f_{0*} = f_{1*}$.

Proof: By assumption, there is a homotopy H: X x [0,1] → Y between f_0 and f_1 such that $H(x_0,t) = y_0$ for all t. It follows that if f is a loop in X based at x_0, then the composition

$$[0,1] \times [0,1] \xrightarrow{(f \times Id)} X \times [0,1] \xrightarrow{(H)} Y$$

is a homotopy between $f_0 \circ f$ and $f_1 \circ f$, and f is a loop at x_0 and H maps $x_0 \times [0,1]$ to y_0.

Theorem: The inclusion map j: $S^n \to R^{n+1} - 0$ induces an isomorphism of fundamental groups.

Proof: Let $X = R^{n+1} - 0$. Let $b_0 = (1, 0, \ldots, 0)$. Let r: $X \to S^n$ be the map $r(x) = x / ||x||$. Then r

o j is the identity map of S^n, so that $r_* \circ j_*$ is the identity homomorphism of $\pi_1(S^n,b_0)$.
Now consider the composition j o r, which maps X into itself:

$$j \circ r:\ X \to S^n \to X.$$

This is not the identity map of X, but it is homotopic to the identity map. Also the homotopy H: X x [0,1] → X, given by

$$H(x,t) = (1-t)x + tx / ||x||,$$

is a homotopy between the identity map of X and the map j o r. It follows from the preceding Lemma that the homomorphism $(j \circ r)_* = j_* \circ r_*$ is the identity homomorphism of $\pi_1(X,b_0)$.

<u>Theorem</u>: Let A be a deformation retract of X. Let x_0 be in A. Then the inclusion map

$$j: (A,x_0) \to (X,x_0)$$

induces an isomorphism of fundamental group.
<u>Proof</u>: The proof is a generalization of the proof of the previous theorem.

<u>Theorem</u>: (Seifert-Van Kampen Theorem)
Let X = U∪V, where U and V are open in X. Assume that U, V, and U∩V are path connected. Let x_0 be in U∩V. Let H be a group, and let

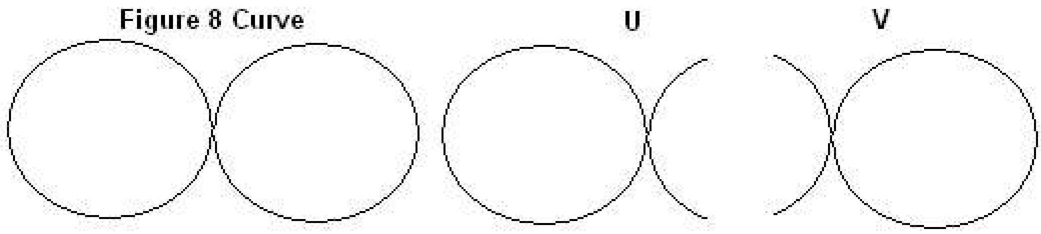
$$\varphi_1: \pi_1(U,x_0) \to H,$$
$$\varphi_2: \pi_1(V,x_0) \to H$$

be group homomorphisms. Let i_1, i_2, j_1, j_2 be the group homomorphisms indicated in the following diagram, each induced by inclusion.

$$\begin{tikzcd}
 & \pi_1(U,x_0) \arrow[d, "j_1"] \arrow[dr, "\varphi_1"] & \\
\pi_1(U \cap V,x_0) \arrow[ur, "i_1"] \arrow[r] \arrow[dr, "i_2"'] & \pi_1(X,x_0) \arrow[r, dashed, "\varphi"] & H \\
 & \pi_1(V,x_0) \arrow[u, "j_2"'] \arrow[ur, "\varphi_2"'] &
\end{tikzcd}$$

If $\varphi_1 \circ i_1 = \varphi_2 \circ i_2$, then there is a unique homorphism $\varphi:\pi_1(X,x_0) \to H$ such that $\varphi \circ j_1 = \varphi_1$ and $\varphi \circ j_2 = \varphi_2$.
Basically the theorem says that if φ_1 and φ_2 are arbitrary homomorphisms that are "compatible on U∩V," then they induce a homomorphism from $\pi_1(X,x_0)$ into H.

<u>Example</u>: (Fundamental Group of The Figure Eight Curve)



Let X be the figure eight curve, and let the sets U, V in X be chosen as in the diagram above. Thenπ_1(U) is isomorphic to Z, and π_1(V) is isomorphic to Z. By the Seifert-Van Kampen Theorem. We conclude that π_1(X) = Z x Z.

The Knot Group

Now we have defined fundamental groups in a topological space, we are going to apply it to the study of knots and use it as an invariant for them.

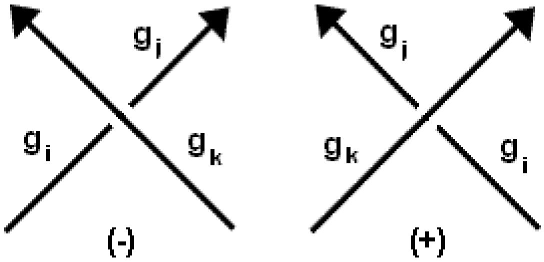
Definition: Two knots K_1 and K_2 contained in R^3 are equivalent if there exists an orientation-preserving homeomorphism h: $R^3 \rightarrow R^3$ such that $h(K_1) = K_2$.

If we compare two equivalent knots, K_1 and K_2, as topological objects, then $R^3\backslash K_1$ is homeomorphic to $R^3\backslash K_2$. Therefore, $R^3\backslash K_1$ and $R^3\backslash K_2$ have isomorphic fundamental groups. In fact, if we can prove that the fundamental groups $\pi_1(R^3\backslash K_1)$ and $\pi_1(R^3\backslash K_2)$ are not isomorphic, then we know that the knots K_1 and K_2 are not equivalent. This is one of the most common methods to distinguish different knots, and the fundamental group $\pi_1(R^3\backslash K)$ is called the *knot group* of K.

Theorem: The knot group is an invariant of ambient isotopy.

Proof: Recall that by definition, two topological objects are equivalent if they are homeomorphic. It is clear that the complements of a knot under ambient isotopy are homeomorphic, and homeomorphic topological spaces have isomorphic fundamental groups. So the knot group is invariant under ambient isotopy.

In 1925, Wilhelm Wirtinger proved that given a knot diagram of a knot with n-crossings, the knot group may be generated by a set of n (homotopy classes of) loops, one for each arc. Let K be a knot, we can write down a presentation of $\pi_1(R^3 \backslash K)$ in the following way: Select an orientation of K. Label the three arcs at a crossing with g_i for distinct i's. For convenience, we're going to label g_i, g_j, g_k as below:



Following the picture above, define the relations between the group generators as the following:

$$(1)\ g_j = g_k\, g_i\, g_k^{-1}\ (-), \qquad (2)\ g_j = g_k^{-1}\, g_i\, g_k\ (+).$$

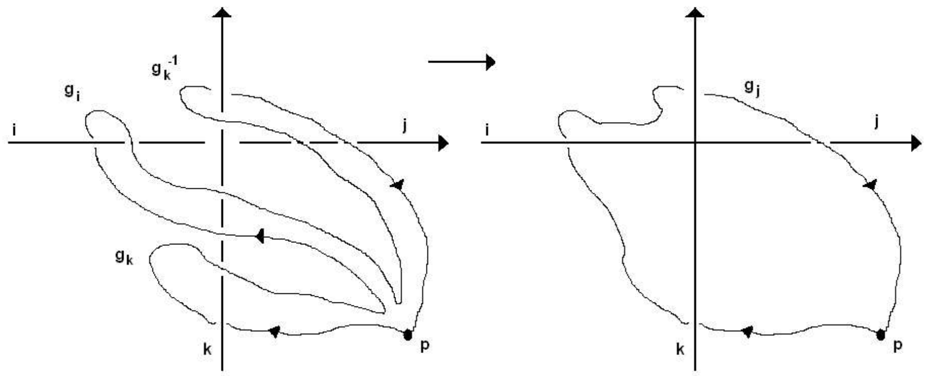
Here (+), (-) denote the sign of the crossing.

The symbol g_i represents the loop that, starting from a base point somewhere in the complement of the knot, goes straight to the i^{th} over passing arc, encircles it in a positive direction (with linking number +1) and returns straight to the base point. The resulting presentation is called the *Wirtinger presentation* of the knot group.

Moreover, it is possible to prove equation (1) and (2), where g_j, g_k, g_i are homotopy classes of the loops described above. The following is the proof for equation (1):

Proof:

$$g_j = g_k\, g_i\, g_k^{-1}\ (-)$$



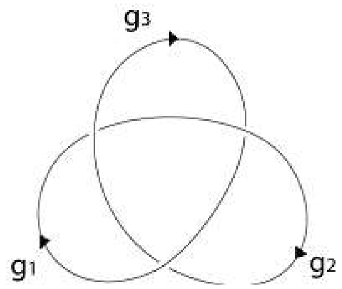
If there are m arcs in the diagram and n crossings, then the group of the link is isomorphic to group G with this presentation:

$$G = \langle g_1, g_2, \ldots, g_m; r_1, r_2, \ldots, r_n \rangle,$$

where G is the quotient of the free group on generators $\{g_1, g_2, \ldots, g_m\}$ by the smallest normal subgroup generated by the relations $\{r_1, r_2, \ldots, r_n\}$.

Example: (Trefoil)

Consider the trefoil:



We get the following three relations:

$$g_2 = g_3\, g_1\, g_3^{-1}$$
$$g_3 = g_1\, g_2\, g_1^{-1}$$
$$g_1 = g_2\, g_3\, g_2^{-1}$$

And the group is given by the presentation:

$$G = \langle g_1, g_2, g_3 \mid g_2 = g_3\, g_1\, g_3^{-1},\ g_3 = g_1\, g_2\, g_1^{-1} \rangle$$

We see here that only two relations are necessary. In fact, we will prove later that a knot with n crossings, at most n – 1 relations are actually needed.

Let Σ_n be the group of permutation of n elements. Define a group homomorphism $\varphi: G \to \Sigma_3$ by:

$$g_1 \to (1,2),\ g_2 \to (2,3),\ g_3 \to (3,1),$$

where (i,j) in Σ_3 denotes the element of the permutation group which switches i and j.
We can check that φ preserves the relations:

$$\varphi(g_3\, g_1\, g_3^{-1}) = (3,1)(1,2)(1,3) = (2,3),$$
$$\varphi(g_1\, g_2\, g_1^{-1}) = (1,2)(2,3)(2,1) = (3,1).$$

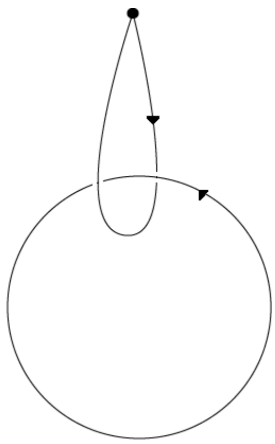
Moreover, one can easily check that φ is an isomorphism. So, we can conclude that G is isomorphic to Σ_3. In particular, G is not abelian.

Next we will prove that the knot group of the unknot is infinite cyclic.

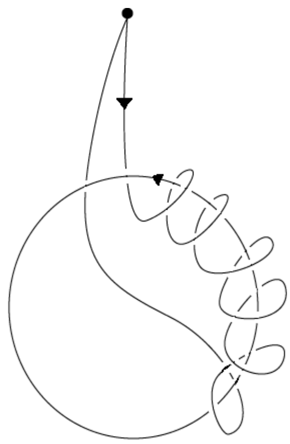
<u>Theorem</u>: The knot group of the unknot is infinite cyclic.

Proof: Let K denote the unknot in R^3, and construct the Wirtinger presentation of the knot group of K. Choose a base point p in $R^3 \backslash K$, then the class of a loop based at p and linking K once is the generator of the group, say x. Any loop based at p which links K is homotopic to some power of x. A loop which does not link K is contractible in $R^3 \backslash K$, hence homotopically trivial. So we can conclude that $\pi_1(R^3 \backslash K, p)$ is the infinite cyclic group generated by x.

` p p



x is a generator of the group



x^6 (The loop based at p isn't contractible.)

Here is another way (using the Seifert-Van Kampen theorem) to prove that the group of the unknot in R^3 is infinitely cyclic.

Theorem: The knot group of the unknot is infinitely cyclic.

Proof: Consider the unknot circle in R^3. The first step is to obtain a decomposition of the 3-sphere S^3 into the following two pieces. Let

$$A = \{(x_1, x_2, x_3, x_4) \text{ in } S^3 \mid x_1^2 + x_2^2 \leqq x_3^2 + x_4^2\},$$

$$B = \{(x_1, x_2, x_3, x_4) \text{ in } S^3 \mid x_1^2 + x_2^2 \geqq x_3^2 + x_4^2\}.$$

It is clear that $A \cup B = S^3$, and that

$$A \cap B = \{(x_1, x_2, x_3, x_4) \text{ in } S^3 \mid x_1^2 + x_2^2 = 1/2 \text{ and } x_3^2 + x_4^2 = 1/2\}.$$

It is clear that $A \cap B$ is a torus. More precisely, it is the Cartesian product of the circles $x_1^2 + x_2^2 = 1/2$ and $x_3^2 + x_4^2 = 1/2$, lying in (x_1, x_2) and (x_3, x_4) planes respectively.

Also A and B are both solid tori. We shall show this by constructing a homeomorphism. First, let

$$D = \{(x_1, x_2) \text{ in } R^2 \mid x_1^2 + x_2^2 \leqq 1/2\} \quad \text{be a disc,}$$

$$S = \{(x_3, x_4) \text{ in } R^2 \mid x_3^2 + x_4^2 \quad 1/2\} \text{ be a circle.}$$

Define a map f: D x S → A by:

$$f(x_1, x_2, x_3, x_4) = (x_1, x_2, 2^{1/2}x_3[1 - (x_1^2 + x_2^2)]^{1/2}, 2^{1/2}x_4[1 - (x_1^2 + x_2^2)]^{1/2}).$$

One can check that this is indeed a homeomorphism. A similar construction holds for the set B. It is clear that the torus $A \cap B$ is the common boundary of the two solid tori A and B.

Now we consider the group of an unknot circle K in S^3. We can take K as the middle circle of A:

$$K = \{(x_1, x_2, x_3, x_4) \text{ in } A \mid x_1 = x_2 = 0\},$$

then K is the unit circle in the (x_3,x_4) plane. The boundary of A is a deformation retract of A\K. Since the boundary of A in $A \cap B$, it follows that B is a deformation retract of S^3\K. It is also clear that the middle circle of B,

$$\{(x_1, x_2, x_3, x_4) \text{ in } B \mid x_3 = x_4 = 0\},$$

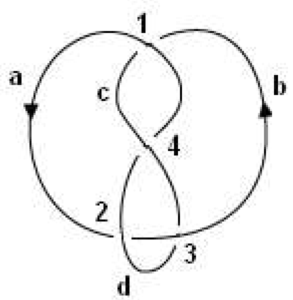
is a deformation retract of B. Therefore, the middle circle of B is a deformation retract of $S^3\backslash K$. Hence $S^3\backslash K$ has the homotopy type of a circle, and the group of K is infinite cyclic, by the fact the fundamental group is preserved under deformation retracts.

Combining the two results from above, we now know that the trefoil and the unknot are not equivalent since they have different knot groups.

Next we will compute the knot group for the figure eight knot.

Example: (Figure Eight Knot)

Consider the figure eight knot:



Let {1, 2, 3, 4} denote crossings, and {a, b, c, d} denote the group generators corresponding to the overpasses.

Since we have 4 over passes, immediately we know there are at most 4 generators in the group. Moreover we can find all relations for this group in the Wirtinger presentation:

1: $c = a^{-1} b a$, 2: $b = d a d^{-1}$,

3: $d = b c b^{-1}$, 4: $a = c^{-1} d c$.

But we only need at most 3 of the relations out of the 4, so we can drop the 4th one out. We get:

$$c = a^{-1} b a,$$
$$b = d a d^{-1},$$
$$d = b c b^{-1} \quad \Rightarrow \quad d b = b c.$$

Next,

$$b = d a d^{-1},$$
$$d b = b a^{-1} b a \quad \Rightarrow \quad b^{-1} d b = a^{-1} b a.$$

Then,

$$a d a^{-1} d a = d a d^{-1} a d.$$

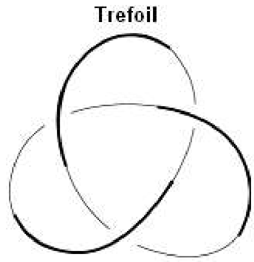
Lastly, we obtain a representation of the knot group for the figure eight knot,

$$G = \langle a, d \mid a d a^{-1} d a = d a d^{-1} a d \rangle.$$

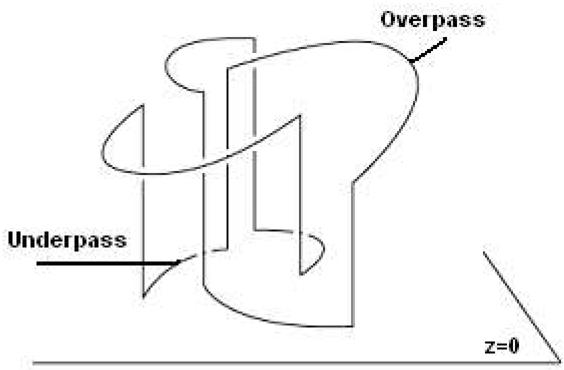
Moreover, since the knot group for the figure eight knot only has 2 generators because we can represent $c = a d^{-1} a d a^{-1}$ and $b = d^{-1} a d$, it is not isomorphic to Σ_3, the knot group of the trefoil. So the figure eight knot is not equivalent to the trefoil.

<u>Theorem</u>: Let R^{3+} be the closed half-space defined by the inequality $z \geqq 0$, and let K be a knot in R^{3+}. Then $\pi_1(R^{3+}\backslash K, p)$ is the free group generated by $\{g_1, g_2, \ldots, g_n\}$, where $\{g_1, g_2, \ldots, g_n\}$ represent the elements corresponding to the overpasses $\{x_1, x_2, \ldots, x_n\}$ of K respectively.

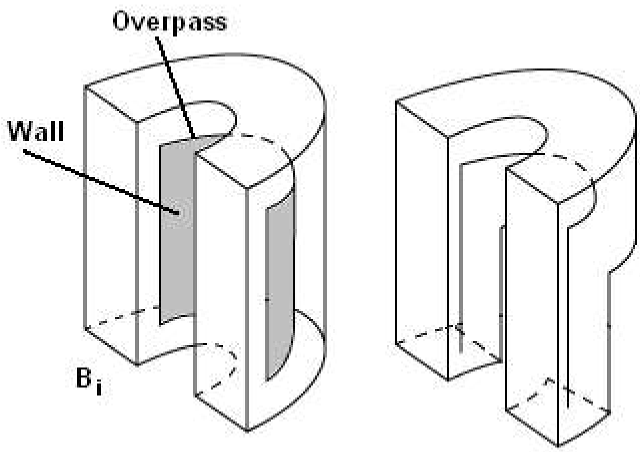
<u>Proof</u>: In the proof, we shall use the trefoil as an example for the figure.



Let $\underline{K}$ denote the union of overpasses of K together with the vertical line segments. Then clearly $R^{3+}\backslash K$ and $R^{3+}\backslash\underline{K}$ have the same fundamental group.



For each over pass we build a vertical wall up from the plane z = 0 to fit exactly underneath the arc, and thicken this wall slightly to get a three-dimensional ball. We do it in such a way that resulting balls $B_1, B_2, \ldots, B_n$ are disjoint.



Suppose we now remove the interior of each B_i, plus the interior of the horseshoe shaped disc in which it meets the plane z = 0. Then clearly the resulting space X is simply connected (i.e., has trivial fundamental group.) We shall build up $R^3\backslash\underline{K}$ as the union $X \cup (B_1\backslash\underline{K}) \cup \ldots \cup (B_n\backslash\underline{K})$.

Any B_i – $\underline{K}$ is homeomorphic to a solid cylinder with its center line removed. This deformation retracts onto a disc minus its center point, and therefore has fundamental group Z generated by a loop which links once around $\underline{K}$. Also, the intersection of $B_i \backslash \underline{K}$ with X is homeomorphic to a disc and is therefore simply connected.

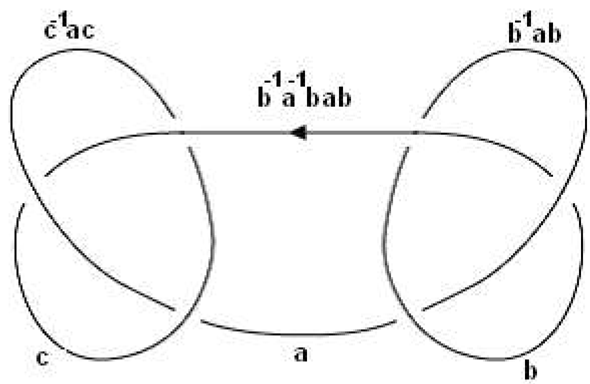
Suppose we know that the fundamental group of X U ($B_1 \backslash \underline{K}$) U … U ($B_i \backslash \underline{K}$) is the free group generated by $g_{1, \ldots,} g_i$, (where $g_1, \ldots, g_i$ are the generators corresponding to arcs $x_1, \ldots, x_i$ respectively.) When we add in $B_{i+1} \backslash \underline{K}$, Van Kampen's theorem tells us that we need an extra generator, which we can clearly take to be g_{i+1}. And by induction, we arrived at the conclusion.

But as powerful as the fundamental group a tool to compare two spaces, it is still not a complete invariant. We will give the following example to show that two different knots that have isomorphic knot groups.

Example: (The Square Knot and The Granny)

The square knot (i.e. the connect sum of a trefoil and its mirror image) can be represented by the following diagram:

The Square Knot



If we take {a, b, c} to be the generators corresponding to the arcs as shown, we can obtain presentations for the other arcs by the set of generators.

$$(b^{-1}\ a\ b)(b^{-1}\ a^{-1}\ b\ a\ b) = (b^{-1}\ a^{-1}\ b\ a\ b)\ b$$

$$\Rightarrow \quad a\ b = b^{-1}\ a^{-1}\ b\ a\ b\ b$$

$$\Rightarrow \quad a\ b\ a\ b = b\ a\ b\ b$$

$$\Rightarrow \quad a\ b\ a = b\ a\ b.$$

Then,

$$(c^{-1}\ a\ c)(b^{-1}\ a^{-1}\ b\ a\ b) = c\ (c^{-1}\ a\ c)$$

$$\Rightarrow \quad c^{-1}\ a\ c\ b^{-1}\ a^{-1}\ b\ a\ b = a\ c$$

$$\Rightarrow \quad a\ c\ b^{-1}\ a^{-1}\ a\ b\ a = c\ a\ c$$

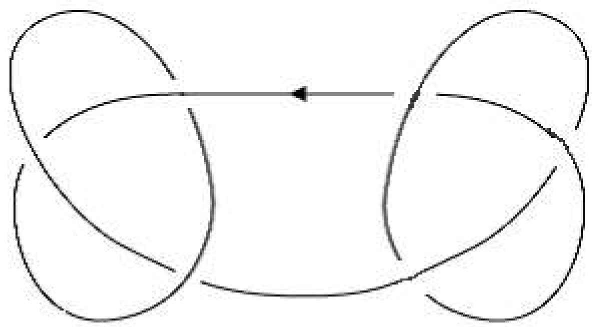
$$\Rightarrow \quad a\ c\ a = c\ a\ c.$$

So the knot group of the square knot is:

$$G = \langle a, b, c \mid a\,b\,a = b\,a\,b,\ a\,c\,a = c\,a\,c \rangle.$$

Similarly, one can check that the granny (i.e. the connect sum of 2 trefoils) has knot group isomorphic to the knot group of the Square knot.

The Granny



Moreover, one can easily check that the square knot and the granny are not equivalent, so this shows that the knot group is not a complete knot invariant.

Conclusion

After establishing the basic knowledge about fundamental groups for a topological space, we see the application of it (the knot group) as a knot invariant. Though the idea of the knot group is not entirely complicated, it is still a very sufficient and powerful invariant for knots as shown in the theorems and examples provided in the paper.

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