Generalized Knot Polynomials and An Application

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ABSTRACT: In this paper we introduce two generalized knot polynomials, the Kauffman and HOMFLY polynomials, show that they are distinct invariants and show that the Jones polynomial is a special case of each. We then use properties of the Kauffman polynomial to prove invariance of the writhe of different alternating diagrams of an alternating link.

Vaughn Jones' discovery of the Jones Polynomial as a knot invariant was a major breakthrough in knot theory. The importance was perhaps not in the polynomial itself but in the method of construction, i.e. the skein relation. The skein relation allows you to take apart the knot crossing by crossing, and keep track of what happens at each stage. With only one variable however, things can get muddled. The basic ideas behind the generalized polynomials we will explore is that two variables keep a better record of how the link was decomposed.

<u>Generalized Polynomials</u>

HOMFLY Polynomial

The HOMFLY polynomial of a link *L* is a two variable polynomial denoted $P_L(a,z)$. It is completely defined for any link by the following three axioms:

1. $a^{-1}P_{L_+} - aP_{L_-} = zP_{L_0}$.
2. The HOMFLY polynomial of the unknot is 1.
3. The HOMFLY polynomial is an invariant of ambient isotopy.

It is easy to prove using these axioms that there is a unique HOMFLY polynomial for any link using induction on the number of crossings of the link (note that the base case for this induction is the second axiom), and an induction on the uncrossing number.

We give the value of the HOMFLY polynomial for some typical simple knots and links:

1. P(unknot) = 1
2. P(n-component unlink)= $[(a+a^{-1})/z]^{n-1}$
3. P(Hopf link) = $a z^{-1} - a^3 z^{-1} + a z$
4. P(trefoil) = $2a^2 - a^4 + a^2 z^2$

Remember that the Jones polynomial was also defined by three axioms:

1. $t^{-1}V_{L_+} - t\,V_{L_-} = (t^{1/2} - t^{-1/2})\,V_{L_0}$.
2. The Jones polynomial of the unknot is 1.
3. The Jones polynomial is an invariant of ambient isotopy.

This looks very similar to the definition of the HOMFLY polynomial. We see that substituting $a=t$, $z= t^{½} - t^{-½}$ in the HOMFLY polynomial, it satisfies the axioms of the Jones polynomial. Since the axioms completely determine the Jones polynomial, we must have $P_L(t, t^{½} - t^{-½}) = V_L(t)$ for all links L.

<u>Claim</u>: All powers of variables appearing in the HOMFLY polynomial of a link *L* are incongruent modulo 2 to the number of components of the link.

Put a little more simply, the claim says that if there are an even number of components, we only see odd powers of *a* and *z* in the HOMFLY polynomial, and vice versa.

Proof:

We use double induction, first on the number of crossings and then on the uncrossing number. Let a link *L* have *n* components. If there are no crossings, then *L* is the *n*-component unlink with HOMFLY polynomial

$$[(a+a^{-1})/z]^{n-1} = z^{-n+1}\sum c_j\, a^{j}\, a^{j-n+1} = z^{-n+1}\sum c_j\ a^{2j-n+1}$$

The exponents that appear in the polynomial are $-n+1$ and $2j-n+1$ where j runs from 0 to n-1. These are all incongruent to n modulo 2.

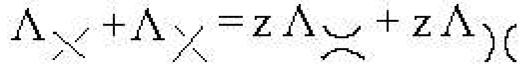
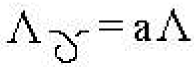
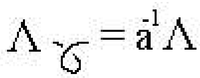
Suppose we know the result for links with up to k-1 total crossings (between separate components and within each component) and any number of components (our base case is true for all numbers of components). Let *L* be a link with *n* components, *k* crossings and uncrossing number one. We use the skein relation of the HOMFLY polynomial on the crossing of *L* which when switched gives us the unknot.

Let L_+ be the link, L_- the link with switched crossing (the unknot since it has uncrossing number one), and L_0 be the link with resolved crossing. Then the skein relation becomes: $a^{-1}P_{L+} - aP_{L-} = zP_{Lo}$, or equivalently $P_{L+} = a^2P_{L-} + a\, zP_{Lo}$. But we know that L_- is the *n*-component unknot. We also know that the resolution as above of a crossing changes the number of components by 1, and decreases the number of crossings by one. Therefore by induction we know that P_{Lo} will have only powers incongruent to $n\pm 1$ modulo 2, and therefore $a\, zP_{Lo}$ will have only powers incongruent to n modulo 2 as desired.

To complete the proof, we do induction on the uncrossing number, which is obtained by an argument almost identical to the above.

Pre-Kauffman Polynomial

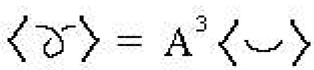
The pre-Kauffman polynomial Λ is a polynomial in two variables *a* and *z* defined on unoriented links and is invariant under only regular isotopy. It is completely defined by the axioms below:

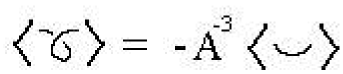
1. 
2. 
3. 
4. Λ (unknot) = 1
5. Λ is invariant under regular isotopy (R2 and R3 moves)

One will note that this is very similar to the Kauffman bracket which is defined by the following:

1. 
2. ⟨0 K⟩ = d ⟨K⟩ where $d = -(A^2 + A^{-2})$
3. ⟨0⟩ = 1

We can derive from these axioms that:





Comparing this with the definition of the pre-Kauffman polynomial, we see that:

$$\langle K \rangle = \Lambda(K)(-A^3, A + A^{-1})$$

Kauffman Polynomial

The Kauffman polynomial is a two variable polynomial, denoted $F_L(a,z)$, defined on oriented links. It is defined as follows:

$$F_L(a,z) = a^{w(L)} \Lambda_L(a,z)$$

As usual *w(L)* is the writhe of *L*. One will notice that this is similar to the definition of the Jones polynomial via the Kauffman bracket:

$$V_L(t) = f_L(t^{1/4}) = -(t^{1/4})^{-3w(L)} \langle L \rangle (t^{1/4})$$

Using the relation we had above between the pre-Kauffman polynomial and the Kauffman bracket we can find a relation between the Kauffman polynomial and the Jones polynomial. Again, we can make the substitution $a = -A^3, z= A + A^{-1}$ so that the pre-Kauffman and Kauffman bracket polynomials are equal. Then we can make a third substitution of $A=t^{-1/4}$, and then we have $a^{w(L)} = -(t^{-1/4})^{3w(L)}$. Under these substitutions we have:

$$F_L(-t^{-3/4}, t^{-1/4} + t^{1/4}) =(-t^{-3/4})^{w(L)} \Lambda_L(-t^{3/4}, t^{-1/4} + t^{-1/4}) = -(t^{1/4})^{-3w(L)} \langle L\rangle(t^{1/4}) = V_L(t)$$

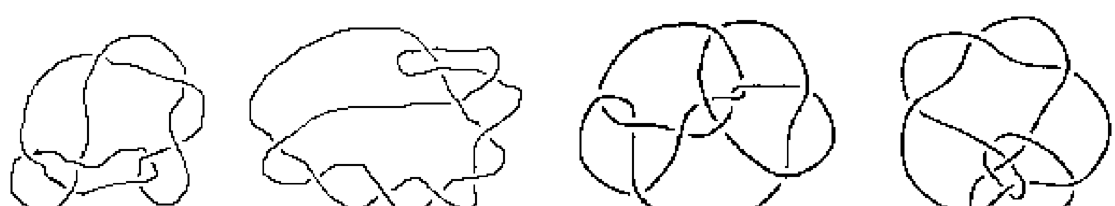
Therefore the Jones polynomial can be obtained as a special case of the Kauffman polynomial.

Here are the values of the Kauffman polynomial on some simple links:

1. F(unknot) = 1
2. F(n-component unlink) = $[(a + a^{-1})/z - 1]^{n-1}$
3. F(Hopf link) = $(-a^3 - a^{-1}) z^{-1} - a^{-2} + (a^3 + a^{-1}) z$
4. F(trefoil) = $-2a^{-2} - a^{-4} + (a^{-3} + a^{-5}) z + (a^{-2} + a^{-4}) z^2$

Distinct Invariants

The following questions are very natural at this point: We had two ways of defining the Jones polynomial, one of which was axiomatic (as in the definition of the HOMFLY polynomial) and one was via a bracket polynomail which was only regular isotopy invariant (as in the definition of the Kauffman polynomial). In the case of the Jones polynomial we got the same thing both times. So how do we know the Kauffman and HOMFLY polynomials are not the same? We can clearly see from the polynomials of some simple links that they are not termwise equal, but could they be similar up to substitution of variables? Another natural question is whether or not either of these polynomials nontrivially extends the Jones polynomial. Does either polynomial give us information that the Jones polynomial alone cannot? The following examples answer both of these questions:



8-8 10-129 11-alternating-79 11-alternating-255

Knots 8-8 and 10-129 have the same HOMFLY polynomial but distinct Kauffman polynomials. Knots 11-alternating-79 and 11-alternating-255 have distinct HOMFLY polynomials but identical Kauffman polynomials:

F(8-8) = $2a^{-5}z - 3a^{-5}z^3 + a^{-5}z^5 - a^{-4} + 4a^{-4}z^2 - 6a^{-4}z^4 + 2a^{-4}z^6 + 3a^{-3}z - 5a^{-3}z^3 + a^{-3}z^7 - a^{-2} + 5a^{-2}z^2 - 9a^{-2}z^4 + 4a^{-2}z^6 + a^{-1}z - 3a^{-1}z^3 + a^{-1}z^5 + a^{-1}z^7 + 2 - z^2 - z^4 + 2z^6 - az + 2az^5 + a^2 - 2a^2z^2 + 2a^2z^4 - a^3z + a^3z^3$

P(8-8) = $- a^{-4} - a^{-4}z^2 + a^{-2} + 2a^{-2}z^2 + a^{-2}z^4 + 2 + 2z^2 + z^4 - a^2 - a^2z^2$

F(10-129) = $2a^{-5}z - 3a^{-5}z^3 + a^{-5}z^5 - a^{-4} + 4a^{-4}z^2 - 6a^{-4}z^4 + 2a^{-4}z^6 + 3a^{-3}z - 5a^{-3}z^3 + a^{-3}z^7 - a^{-2} + 5a^{-2}z^2 - 9a^{-2}z^4 + 4a^{-2}z^6 + a^{-1}z - 3a^{-1}z^3 + a^{-1}z^5 + a^{-1}z^7 + 2 - z^2 - z^4 + 2z^6 - az + 2az^5 + a^2 - 2a^2z^2 + 2a^2z^4 - a^3z + a^3z^3$

P(10-129) = $-a^{-2} - a^{-2}z^2 + 2 + 2z^2 + z^4 + a^2 + 2a^2z^2 + a^2z^4 - a^4 - a^4z^2$

F(11-alternating-79) = $a^{-2}z^2 - 2a^{-2}z^4 + a^{-2}z^6 - a^{-1}z + 5a^{-1}z^3 - 9a^{-1}z^5 + 4a^{-1}z^7 + 2 - 5z^2 + 11z^4 - 16z^6 + 7z^8 - 3az + 10az^3 - 9az^5 - 6az^7 + 6az^9 + 3a^2 - 19a^2z^2 + 44a^2z^4 - 45a^2z^6 + 14a^2z^8 + 2a^2z^{10} - 5a^3z + 14a^3z^3 - a^3z^5 - 20a^3z^7 + 13a^3z^9 + 3a^4 - 19a^4z^2 + 47a^4z^4 - 48a^4z^6 + 17a^4z^8 + 2a^4z^{10} - 5a^5z + 16a^5z^3 - 14a^5z^5 - 2a^5z^7 + 7a^5z^9 + a^6 - 5a^6z^2 + 11a^6z^4 - 16a^6z^6 + 10a^6z^8 - 2a^7z + 6a^7z^3 - 12a^7z^5 + 8a^7z^7 + a^8z^2 - 5a^8z^4 + 4a^8z^6 - a^9z^3 + a^9z^5$

P(11-alternating-79) = $2 + 3z^2 + 3z^4 + z^6 - 3a^2 - 9a^2z^2 - 10a^2z^4 - 5a^2z^6 - a^2z^8 + 3a^4 + 8a^4z^2 + 7a^4z^4 + 2a^4z^6 - a^6 - 2a^6z^2 - a^6z^4$

F(11-alternating-255) = $- 2a^{-2}z^4 + a^{-2}z^6 + 3a^{-1}z^3 - 9a^{-1}z^5 + 4a^{-1}z^7 + 2 - 8z^2 + 19z^4 - 22z^6 + 8z^8 + az - 6az^3 + 15az^5 - 19az^7 + 8az^9 + 3a^2 - 22a^2z^2 + 50a^2z^4 - 42a^2z^6 + 9a^2z^8 + 3a^2z^{10} + a^3z - 13a^3z^3 + 38a^3z^5 - 40a^3z^7 + 16a^3z^9 + 3a^4 - 20a^4z^2 + 44a^4z^4 - 39a^4z^6 + 11a^4z^8 + 3a^4z^{10} - a^5z + 2a^5z^3 + a^5z^5 - 9a^5z^7 + 8a^5z^9 + a^6 - 5a^6z^2 + 10a^6z^4 - 16a^6z^6 + 10a^6z^8 - a^7z + 5a^7z^3 - 12a^7z^5 + 8a^7z^7 + a^8z^2 - 5a^8z^4 + 4a^8z^6 - a^9z^3 + a^9z^5$

P(11-alternating-255) = $2 + 3z^2 + 3z^4 + z^6 - 3a^2 - 9a^2z^2 - 10a^2z^4 - 5a^2z^6 - a^2z^8 + 3a^4 + 8a^4z^2 + 7a^4z^4 + 2a^4z^6 - a^6 - 2a^6z^2 - a^6z^4$

This shows that the two invariants are distinct since each can distinguish a pair of links that the other cannot. Since F(8-8) = F(10-129) we must have V(8-8) = V(10-129) (and similarly for the others), so both the HOMFLY and Kauffman polynomials give strictly more information than the Jones polynomial.

Application

We will now use some properties of the Kauffman and pre-Kauffman polynomials to prove the famous result that the writhe of a reduced alternating diagram is an invariant.

First we need some definitions. A **prime** link is a link not representable as the connected sum of two or more links, neither of which is the one component unlink. It has been proven that every knot is (up to position and ambient isotopy) uniquely decomposable into the direct sum of prime knots. A **bridge** is simply an arc in a planar link diagram, i.e. a single connected line segment. We will mostly be concerned with arcs in link diagrams that form the overcrossing strand in multiple crossings. We call it a bridge because it is forming a bridge over a section of the knot by going over multiple other strands. The **length of a bridge** is the number of overcrossings formed by a bridge. To any link diagram, we can assign a **bridge** length. This is the length of the longest bridge in the diagram. We are ultimately interested in alternating link diagrams; diagrams which alternate over-under as we travel along the knot. Any alternating knot diagram has bridge length 1. For techical reasons we will see later, we need to define **improper**

bridges. They are the four types of bridges shown below that can be removed by ambient isotopy:

improper bridges

Finally, a **centered polynomial** is a Laurent polynomial whose highest x degree and highest x^{-1} degree are equal; e.g. $x^{-4} + 6x^{-1} + 5x^3 + 2x^4$ is a centered polynomial in x.

We are now ready to discuss and prove the theorem.

Theorem: The writhe of a reduced alternating diagram of an alternating link is invariant.

<u>Proof:</u>

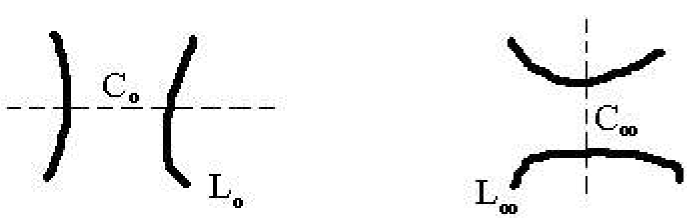
The proof will be made up of several lemmas, many of which are interesting in their own right. We will piece everything together as we go along until the proof is complete.

Lemma 1: If an unoriented link diagram, L, is prime, connected and alternating, then both of the link diagrams formed by the splicings of that diagram at any crossing, L_o and L_∞, are connected and alternating, and at least one of the spliced diagrams is prime.

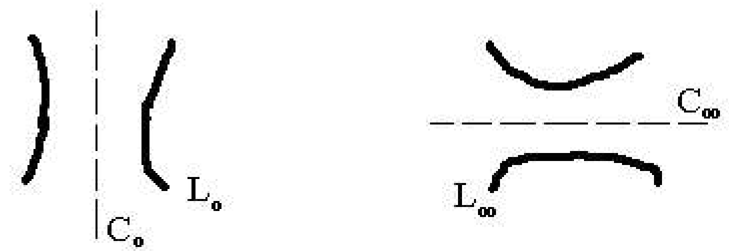
This lemma basically says that at least one of the splicings of a prime link diagram is prime. The proof will be very geometric but the argument is simple.

<u>Proof:</u>

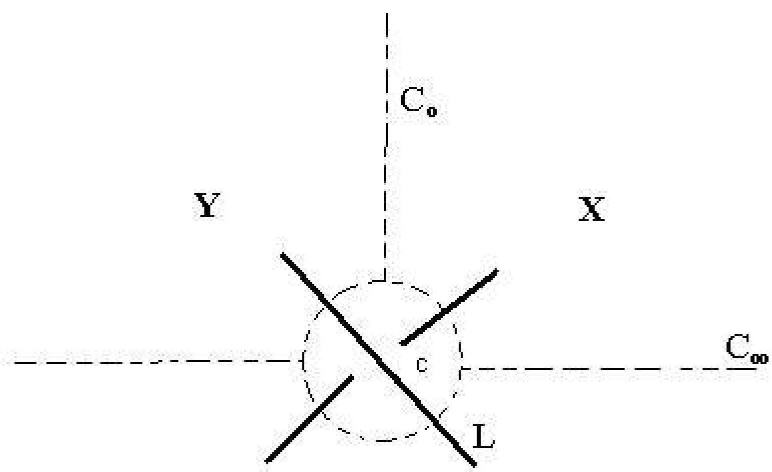
Suppose this lemma were not true, i.e. that both L_o and L_∞, the splicings of L at a crossing c, are composite links. Then we can draw lines through the diagrams of L_o and L_∞ such that away from c each line only intersects the diagram twice. It is clear that the line must go through where the splicing happened or else L was composite to begin with. Suppose the dissecting lines C_o and C_∞ were as in the following diagram:



The diagram to each side of each line is not trivial (i.e. a single arc) or else the crossing c could have been removed with a Reidemeister 1 move. Since C_o and C_∞ only intersect the diagram twice each, we must have that the lines do not intersect the diagram anywhere else. But away from the crossing c the diagrams of L_o and L_∞ are the same as that of L so that C_o and C_∞ do not intersect L away from c. This means that the lines dissect L into the direct sum of two nontrivial knot diagrams. Therefore the lines C_o and C_∞ must be as below:



We can go back and superimpose the lines C_o and C_∞ onto the diagram of L and we can study the regions X and Y:



The circular part of the boundary of X and Y is such that it only intersects L in the four places shown. Since L is a connected link diagram, it must intersect the boundary of the regions X and Y an even number of times (we can of course assume that the diagram is not tangent to the boundaries of X and Yanywhere). Also C_o and C_∞ intersect the diagram twice, so by symmetry we can assume L intersects C_o at least once on the boundary of X and C_∞ at least once on the boundary of X. Since L intersects the boundary of X once on the circular edge of X near the crossing c, we have at least three intersections between L and the boundary of X. However we must have an even number of intersections so there must be a fourth on (without loss of generality) the C_∞ boundary of X. This gives us a total of two intersections between L and C_∞ and hence all such intersections. Examining the region Y, L cannot intersect the circular or the C_∞ part of the boundary. We have at least one intersection between L and the C_o part of the boundary of Y, giving us two intersections between L and the boundary of Y. Any more would have to occur on C_o, and they would have to occur in pairs, but C_o can intersect L at most once more. Therefore L intersects the boundary of Y exactly two times and Y dissects L into the direct sum of two nontrivial links, but this is a contradiction because we said L was prime. Therefore at least one of L_o, L_∞ is prime, and Lemma 1 is proven.

We are now going to examine the quantities N and N-B, where N is the number of crossings of a link diagram and B is its bridge length. First we see how these two quantities vary when we remove improper bridges (see definition of improper bridge for diagrams). One can easily see that in each case if we remove the obvious unnecessary k crossings, then the bridge we see in the picture decreases in length by k. But there may be other bridges in the knot longer than B-k, so B decreases by at most k. Therefore when removing improper bridges, N decreases (strictly) and N-B decreases or remains constant. The important fact we will use later is that if a bridge is not improper, then it does not terminate by going under itself.

Lemma 2: The z degree of F_L and Λ_L (these quantities are the same) is less than or equal to the number of crossings N minus the bridge length B. In symbols:

$$deg_z F_L = deg_z \Lambda_L \leq N - B$$

Proof:

Let us look at the set of all link diagrams L that do not satisfy the desired inequality, i.e. links L such that $deg_z F_L > N - B$. We can then take a link L in the set with minimal crossing number N. Suppose L had an improper bridge. Then we can remove the improper bridge to obtain a link diagram L'. Since they are ambient isotopic to each other $F_L = F_{L'}$, and in particular, $deg_z F_L = deg_z F_{L'}$. We know L' has strictly fewer crossings than L, but also by our argument above N'-$B' < N$-B, and hence

$$deg_z F_{L'} > N' - B'$$

i.e. L' is in our set of links and has strictly less than N crossings. This is a contradiction since L was taken to have minimal crossing number in the set. So any link diagram L contradicting our lemma will have no improper bridges.

Now of all links L for which $deg_z F_L > N - B$ which have the minimal crossing number N, take the one with maximum bridge length B. We now use the skein relation for the pre-Kauffman polynomial to resolve L at one of the endpoints (a crossing) of a bridge b in L attaining length B:

$$\Lambda_{L+} + \Lambda_{L-} = z\,(\Lambda_{Lo} + \Lambda_{L\infty})$$

where $L_+ = L$. The maximal bridge length for any link with N crossings violating our inequality is B. Since the crossing we are resolving terminated the bridge b, the link diagram L- has bridge length at least B+1, and N crossings. Therefore L- must satisfy

$$deg_z F_{L-} = deg_z \Lambda_{L-} \leq N - B - 1$$

So by the skein relation, at least one of Λ_{Lo}, $\Lambda_{L\infty}$ must have a nonzero term of z to a power strictly greater than N-B-1 = (N-1) – B, i.e without loss of generality

$deg_z \Lambda_{Lo} > (N\text{-}1) - B$. But since each of them has the crossing which terminated b resolved, they each must have bridge length at least B. Therefore if N_0 and B_0 are the crossing number and bridge length of L_0, then:

$$deg_z \Lambda_{Lo} > N_0 - B_0$$

So L_0 is in the set of links which violate our degree inequality, but it has crossing number $N_0 = N - 1 < N$ and this contradicts the minimality of N in the set. Therefore the set must be empty and our inequality:

$$deg_z F_L = deg_z \Lambda_L \leq N - B$$

always holds, and Lemma 2 is proven.

Lemma 3: If a link projection L is prime, connected and alternating then the coefficient of z^{n-1} in Λ_L (this is the highest power of z appearing) is a centered polynomial in a with positive first and last coefficients, i.e. of the form:

$$c_{-m}\, a^{-m} + c_{-m+1}\, a^{-m+1} + \ldots + c_0 + \ldots + c_{m-1}\, a^{m-1} + c_m\, a^m$$

where $c_{-m} > 0,\ c_m > 0.$

<u>Proof:</u>

We prove this easily by induction on the crossing number N. If $N = 0$, then the link is the 1-component unlink since we are considering connected diagrams. The theorem is trivially satisfied. Suppose the theorem is true for prime, connected, alternating links with any number of crossings less than N, and let L be a prime connected alternating link with N crossings. Pick any crossing and resolve the link diagram using the skein relation:

$$\Lambda_{L+} + \Lambda_{L-} = z\,(\Lambda_{Lo} + \Lambda_{L\infty})$$

where $L_+ = L$. Since L was alternating, it has bridge length 1. But when we switch a crossing to get L-, we create a bridge of length three. Therefore $deg_z \Lambda_{L-} \leq N - 3$ and it does not contribute to the z^{n-1} coefficient of Λ_L. We know that one of Λ_{Lo}, $\Lambda_{L\infty}$ is prime, and both are connected alternating. So without loss of generality, the theorem holds for Λ_{Lo}, i.e. it has highest z degree n-2, and its coefficient is a centered polynomial in a with positive first and last coefficient. Now we need to look at $\Lambda_{L\infty}$. If L_∞ is prime, then $\Lambda_{L\infty}$ satisfies the same conditions as Λ_{Lo}. In this case the coefficient of z^{n-1} in Λ_L is the sum of the z^{n-2} coefficients of Λ_{Lo} and $\Lambda_{L\infty}$ which will again be a centered polynomial in a with positive first and last coefficients and we are done. So we need to address the case when L_∞ is not prime. We know $deg_z F_L = deg_z \Lambda_L$, and that F_L is multiplicative under direct summation of knots. So let $L_\infty = L_1 \# L_2 \# \ldots \# L_k$. Then:

$$deg_z F_{L\infty} = deg_z F(L_1) \bullet deg_z F(L_2) \bullet \ldots \bullet deg_z F(L_k)$$

If the link L_i has crossing number N_i and bridge length B_i then we know:

$$deg_z F(L_1) = deg_z \Lambda (L_1) \leq N_i - B_i$$

Using that $1 \leq B_i$ for all links L_i and $N_1 + N_2 + \ldots + N_k = N - 1$, we know:

$$deg_z F_{L\infty} < N - 2$$

and therefore will not contribute to the z^{n-1} coefficient of Λ_L and Lemma 3 is proven.

We now have all the necessary knowledge to prove the theorem.

Proof of Theorem:

The Kauffman polynomials of any two reduced, prime, alternating projections L_1 and L_2 of the same link L are equal, so from the definition of the Kauffman polynomial we have:

$$F(L_1)(a,z) = a^{w1} \Lambda(L_1)(a,z) = F(L_2)(a,z) = a^{w2} \Lambda(L_2)(a,z)$$

In particular, the pre-Kauffman polynomials of the two projections differ by a factor of a^{w1-w2} where w_1 and w_2 are the writhes of each projection. Since the leading coefficient of both pre-Kauffman polynomials is a centered polynomial in a, then we must have $a^{w1-w2} = 1$. Multiplying a centered polynomial in a by any nonzero power of a will result in a polynomial that is not centered. Hence $w_1 - w_2 = 0$, i.e. the two projections have the same writhe.

The theorem is poven for prime alternating links. The general case (composite links) follows from the prime case since the writhe is additive under direct summation and knots are uniquely decomposable into primes. Therefore the full theorem is proven.

The story doesn't end with the Kauffman and HOMFLY polynomials. Until a complete link invariant is discovered, it never will. A lot of current research on knot polynomials is focused on the Colored Jones Polynomial which uses quantum invariants and is much beyond anything in this paper. The curious reader will find many resources about the Colored Joned Polynomial online.

Bibliography:

Lickorish, W. B. Raymond; An Introduction To Knot Theory; Springer-Verlag New York Inc.; October 1997.

Kauffman, Louis H.; "New invariants in the theory of knots"; *Amer. Math. Monthly* 95 (1988), no. 3, 195--242.

Kidwell, Mark E.; "On the degree of the Brandt-Lickorish-Millett-Ho Polynomial of a link"; *Proc. Amer. Math. Soc.*, Vol. 100, No. 4, August 1987.

Thistlethwaite, Morwen B.; "Kauffman's Polynomail and Alternating links"; *Topology*, Vol. 27, No. 3, pp 311-318. 1988.