A Brief History of Knot Theory

Erin Colberg

Abstract: This paper will provide a chronological description of the development of Knot Theory. It will show how Knot Theory originated from early attempts to apply the study of knots to other disciplines, and over time has emerged as a promising field of mathematical exploration in its own right. The paper will enumerate the prominent mathematicians and scientists whose observations and discoveries developed Knot Theory from the 18th century to the present. Our exploration of the history of Knot Theory will conclude with a discussion of current conjecture regarding possible applications and the direction of the future of Knot Theory.

I. Introduction

Knot Theory as we know it first gained prominence as a physicist's erroneous concept of a model for the atom. It was kept alive by the efforts of a few diligent physicists until the twentieth century when mathematicians took up the challenge. Today, mathematical theories regarding knots are being applied to the fields of physics, biology, and chemistry.

II. Early Work Related to Knots

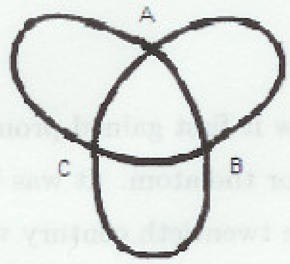
Perhaps the first reference to knots from a mathematical perspective comes from 18th century French mathematician Alexandre-Theophile Vandermonde (1735-1796). In 1771, he opened a paper titled "Remarques sur les problemes de situation," with this statement:

> Whatever the twists and turns of a system of threads in space, one can always obtain an expression for the calculation of its dimensions, but this expression will be of little use in practice. The craftsman who fashions a braid, a net, or some knots will be concerned, not with questions of measurement, but with those of position: what he sees there is the

manner in which the threads are interlaced. [13]

The Work of Gauss; The Gauss Linking Number

While there was much conjecture among certain 18th century mathematicians that knots could be viewed as mathematical entities, it was not until the 19th century that Carl Friedrich Gauss (1777-1855) made the first inroads toward the study of what we now refer to as Knot Theory. One of the oldest notes found among Gauss' belongings was a collection of knot drawings dated 1794. Gauss created a method for the tabulation of knots in which he drew the universe of the knot, labeled the crossings, then chronicled the sequence of letters one would encounter if one were to travel from an arbitrary starting point on the knot, around each arc and back to the starting point. A knot with n crossings would be classified by a sequence of $2n$ letters, called the "scheme of the knot." For example, the trefoil would be recorded as "ABCABC" [17].

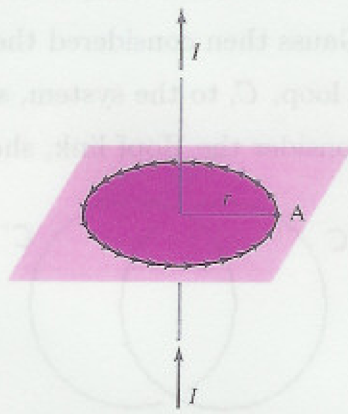


It appears as though Gauss was intrigued by the study of knots, but it was not until his later work with electrodynamics, that he was actually able to apply mathematical concepts to knots, and to derive an important early result in Knot Theory. In 1833, Gauss wanted to know how much work was done on a magnetic pole moving along a closed curve in the presence of a loop of current. He considered two non-intersecting loops, α and β. On January 22, 1833, he answered his question, and in the process discovered what is now referred to as the "Gauss Linking Number" [10].

In his derivation of the Linking Number, he compared two alternate approaches to the magnitude of the magnetic field produced by a current flowing through a closed loop of wire. Gauss began with the magnetic field, **B**, produced by a current, I, passing through a closed loop of wire, A, expressed using Ampere's Law:

$$\oint_A \mathbf{B} \cdot \mathbf{dl} = \mu_0 I,$$

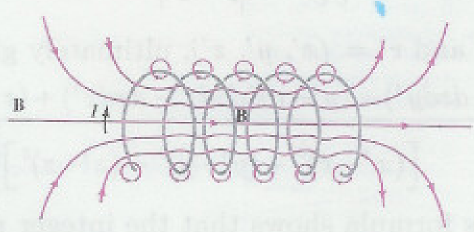
in which μ_0 is the permeability of free space [4,10]. This formula is based on the assumption that the current carrying wire passes through the enclosed surface exactly once, as in the diagram.



Using Ampere's Law to obtain the magnetic field for a solenoid, Gauss generated a formula for the magnetic field in a system in which the current carrying wire passes through the enclosed surface m times:

$$\oint_A \mathbf{B} \cdot \mathbf{dl} = \mu_0 m I,$$

for some integer m. For example, in the diagram below, one can assume that the coil continues out of the scale of the diagram, so that it crosses the plane of the paper m times [4,10].



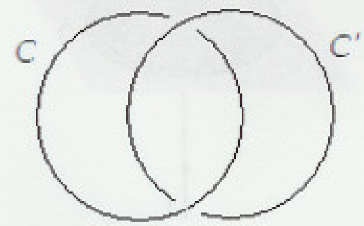
Gauss then derived the magnetic field in an identical system using a different technique. This method, called the Biot-Savart Law, states that

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi} \frac{\mathbf{dl} \times \hat{\mathbf{r}}}{r^2},$$

in which $d\mathbf{B}$ is the contribution of the total magnetic field due to an infinitesimal length, $\mathbf{dl}$, along the current carrying wire, and $\hat{\mathbf{r}}$ is the unit vector along the direction of the displacement vector from $\mathbf{dl}$ to the point of reference [4,10]. He considered a closed loop of wire, C', carrying a current, I. Integrating around C' he obtained that

$$\mathbf{B}(\mathbf{r}) = \frac{-I\mu_0}{4\pi}\int_{C'}\frac{(\mathbf{r}-\mathbf{r}')\times d\mathbf{l}'}{|\mathbf{r}-\mathbf{r}'|^3},$$

in which **r** and **r'** are displacement vectors from the differential element, **dl'** to the point of reference [10]. Gauss then considered the effect if he introduced a second, non-current carrying loop, C, to the system, such that C and C' are non-intersecting. For example, consider the Hopf link, shown below:



To calculate the magnetic field induced in this second loop, Gauss again used the Biot-Savart Law, this time integrating over the second loop, C, to obtain

$$\int_C \mathbf{B}\bullet d\mathbf{l} = \frac{-I\mu_0}{4\pi}\int_C\int_{C'}\frac{(\mathbf{r}-\mathbf{r}')\times d\mathbf{l}'\bullet d\mathbf{l}}{|\mathbf{r}-\mathbf{r}'|^3}.$$

Combining the formulas he obtained using Ampere's Law and the Biot-Savart Law, Gauss arrived at the following relation:

$$-\frac{1}{4\pi}\int_C\int_{C'}\frac{(\mathbf{r}-\mathbf{r}')\times d\mathbf{l}'\bullet d\mathbf{l}}{|\mathbf{r}-\mathbf{r}'|^3} = m.$$

Gauss let $\mathbf{r} = (x, y, z)$, and $\mathbf{r}' = (x', y', z')$, ultimately giving him

$$-\frac{1}{4\pi}\iint\frac{(x'-x)(dydz'-dzdy')+(y'-y)(dzdx'-dxdz')+(z-z')(dxdy'-dydx')}{\left[(x'-x)^2+(y'-y)^2+(z'-z)^2\right]^{\frac{3}{2}}} = m.$$

Gauss realized that this formula shows that the integer m is actually the Linking Number of the loops ***C*** and ***C'*** [2,10]. He wrote, "A main task (that lies) on the border between *geometria situs* and *geometria magnitudinis* is to count the windings of two closed or infinite lines. . . m is the number of windings. This value is shared, i.e., it remains the same if the lines are interchanged," [10]. He was intrigued by this discovery, and went on to prove that the Gauss Linking Number does not change under smooth deformation of loops α and β. The Gauss Linking Number is therefore unchanging under ambient isotopy; it became the earliest discovered link invariant. The Gauss Linking Number was a critical discovery, as it was the first method developed to distinguish two non-equivalent links from each other [2,10].

The Work of Listing

Gauss' work inspired other mathematicians to pursue Knot Theory. Johann Benedict Listing (1808-1882) studied under Gauss at Göttingen in the 1830's, and became interested in knots during his study of topology. Listing's interests were varied; he is credited with the first usage of the term "topologie" in 1836, and was the first to observe and document the properties of the "Möbius band," four years before August Ferdinand Möbius [13]. In pursuing his own studies, Listing modified Gauss' knot notation. He colored each region on a knot diagram black or white, then logged the number of regions of each color bounded by different numbers of arcs. It later became clear that this notation was unsound, as one is unable to construct a unique diagram from Listing's knot notation [17]. In 1847, Listing published "Vorstudien zur Topologie," a paper partially devoted to the study of knots. Listing's specific interest was the chirality of knots, or the equivalence of a knot to its mirror image. The significant result included in his paper was the statement that the right and left trefoil are not equivalent, or not amphichiral, although this was not actually proven until 1914 by Max Dehn (1878-1952) using newly published information about the knot group. Later, Listing stated that the figure eight knot and its mirror image are equivalent, or amphichiral [13].

The Work of Thomson

Knot Theory was attracting the attention of physicists as well as mathematicians. The most prominent of these was English physicist, Sir William Thomson (Lord Kelvin) (1824-1907). During the 1860's, the scientific world was divided into two groups: those who supported "corpuscular theory," (the theory that matter is composed of atoms), and those who supported the theory that matter consisted of waves. Thomson was attempting to develop a new theory that combined these two ideas [13,16].

The work of physicist Hermann von Helmholtz (1821-1894) presented a foundation for what would be Thomson's Theory of Vortex Atoms. In 1858, a paper had been written by Helmholtz titled "On the Integrals of Hydrodynamic Equations to Which Vortex Motions Conform," involving, among other concepts, the idea that there exists an all permeating medium which he called the "ether." Helmholtz analyzed the idea of this theoretical ether, and concluded that the vortices of ether, an ideal fluid, were stable. It followed that these stable vortices could become knotted and still retain their original identities [3,4].

Building upon this foundation, Thomson theorized that matter is composed of "vortex atoms," or three-dimensional knotted tubes of ether. He proposed that different twisting and crossing formations corresponded to different elements. Atoms could be classified by the knots that they resembled, and the representative knot would help to establish some physiochemical properties of the atom. These vortex atoms existed at different energy levels determined by their frequency of vibration. According to the vortex atom theory, molecules could be thought of as intertwined vortex atoms, which would resemble links [13,14,16].

The Work of Maxwell

A friend of Thomson, physicist James Clerk Maxwell (1831-1879), was interested in the idea that knots could be used in the study of electricity and magnetism. He wrote "A Treatise on Electricity and Magnetism" in 1873, employing the ideas of Gauss in relating knots to physics. He also wrote several (unpublished) papers devoted to the study of knots and links, and in the course of his work, rediscovered the Gauss Linking Number integral. He created knot diagrams in which he specified over and under crossings, and then considered how one could change the diagram without affecting the knot. Maxwell analyzed a region bounded by three arcs, and wrote, "In the first case any one curve can be moved past the intersection of the other two without disturbing them. In the second case this cannot be done and the intersection of two curves is a bar to the motion of the third in that direction," [11]. Maxwell defined the three Reidemeister moves that would be named in the 1920's. Despite the impressive volume of work he did and the numerous attempts by his close friend Peter Tait to convince Maxwell to submit his writings to the Royal Society of Edinburgh, Maxwell's papers were not published until more than a century later [11].

The Works of Tait, Kirkman, and Little

With Thomson's theory of vortex atoms came the need for a system of classification of knots. Physicist Peter Guthrie Tait (1831-1901) began making the first table of knots in 1867. Tait was quoted as saying, "I was led to the consideration of the form of knots by Sir W. Thomson's Theory of Vortex Atoms, and consequently the point of view which, at least at first, I adopted was that of classifying knots by the number of their crossings...". Later, in a report to the British Association for the Advancement of Science, Tait wrote, "The development of this subject promises absolutely endless work - but work of a very interesting and useful kind - because it is intimately connected with the theory of knots, which (especially as applied in Sir W. Thomson's Theory of Vortex Atoms) is likely soon to become an important branch of mathematics," [13]. Although Tait is often recognized for his early tabulation of knots, it was mathematician Thomas Kirkman (1806-1895) who made the first major contribution to the task of classifying knots.

Thomas Kirkman was almost solely interested in knot tabulation for alternating knots, and made a table of diagrams for alternating knots with up to eleven crossings. While some duplicate diagrams were discovered by Tait, this was still a significant early development. During his quest to classify different knots, Kirkman realized that he would have to reduce his knot diagrams in order to minimize the number of duplicates, so he devised a method that involved an operation similar to the second Reidemeister move to simplify his diagrams, thus resulting in a more accurate table of knots [12,17].

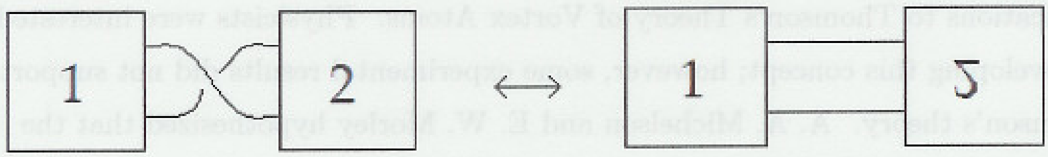
Tait partnered with Charles Newton Little (1858-1923), a professor at the State University of Nebraska, to continue Kirkman's work in the tabulation of knots. They experimented with the different methods of notation, including Listing's notation and Gauss' "scheme of the knot," and eventually decided on a slightly different version of Listing's notation which they altered to eliminate ambiguity. Since there was no way to tell if the knots were equivalent, except by visual examination, their task was difficult. Tait wrote, "... though I have grouped together many widely different but equivalent forms, I cannot be *absolutely* certain that all those groups are essentially different from one another," [17]. Tait and Little eventually discovered some repeated knots in

Kirkman's table, and after some modification to the knot diagrams, they published the first official table of alternating knots with up to ten crossings [16,17].

After his collaboration with Tait produced the table of alternating knots, Little began attempting to classify non-alternating knots. Although non-alternating knots do not exist with fewer than eight crossings, this endeavor would prove to be much more challenging than the tabulation of alternating knots because of the sheer number of non-alternating diagrams that can be produced. A given knot projection with n crossings has two possible alternating diagrams: one can be obtained by selecting a crossing and designating it an over-crossing, then completing the diagram such that the crossings alternate; the other can be obtained by the same process, designating the chosen crossing an under-crossing. Meanwhile, the same n-crossing knot projection has 2^n-2 possible non-alternating diagrams. During his work with the tabulation of non-alternating knots, Little developed what he assumed, incorrectly, was a knot invariant similar to the absolute value of the writhe. In 1899, after six years of work, Little published a table of forty-three ten-crossing, non-alternating knots, including 551 variations of the already classified diagrams [13,16,17].

Tait's Conjectures

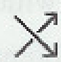
Of the early knot theorists, Tait's contributions were varied and significant. He was curious about the "beknottedness" of a knot, or the existence of the invariant that we now know as the unknotting number; he also created a list of all the amphicheiral knots up to ten crossings. As a result of his work with Little on classifications of knots, Tait became interested in the properties of reduced knot diagrams, and how to obtain them. He defined a "nugatory" crossing as a crossing that divided a diagram into two non-intersecting parts, as in the diagram:



Tait stated that a nugatory crossing could be added or removed from a diagram by a twist, what we now know as the first Reidemeister move, also shown in the diagram above. Tait's three conjectures are as follows:

1. *"An alternating diagram with no nugatory crossings, of an alternating link realizes the minimal number of crossings among all diagrams representing the link,"* [13].

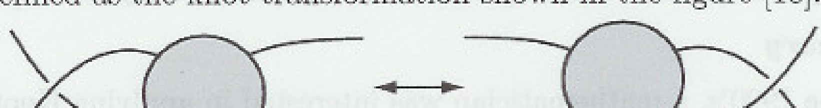
He "proved" this first conjecture simply by showing that the removal of a nugatory crossing was possible, and that it reduced the number of crossings in the diagram. Tait's second conjecture followed from his first, and states that,

2. *"Two alternating diagrams, with no nugatory crossings, of the same oriented link have the same Tait (or writhe) number, i.e. the signed sum of all crossings of the diagram with the convention that*  *is +1 and*  *is -1,"* [13].

Tait's third conjecture states that

3. *"Two alternating diagrams, with no nugatory crossings, of the same link are related by a sequence of flypes,"* [13].

A flype is defined as the knot transformation shown in the figure [18]:



Tait provided non-rigorous proofs for his conjectures, which were only recently proven in the 1980's following the discovery of the Jones polynomial. Tait published his findings over the course of nine years in three papers titled "On Knots I," "On Knots II," and "On Knots III" [13].

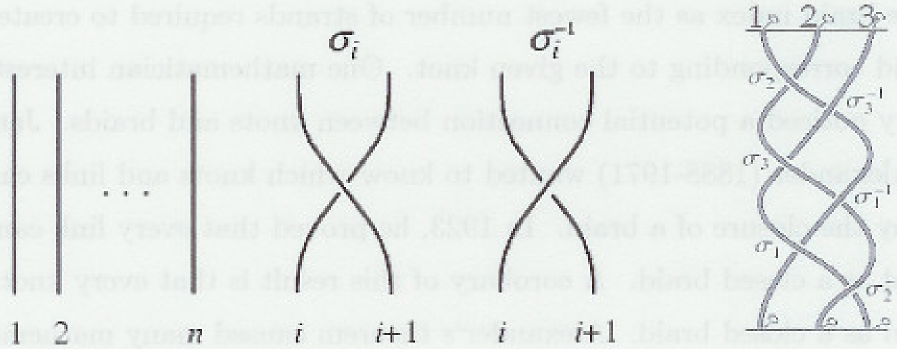
The Virtual Demise of Early Knot Theory

Mathematicians were quickly becoming interested in Knot Theory as a new area of study, but it was still considered significant primarily due to its applications to Thomson's Theory of Vortex Atoms. Physicists were interested in developing this concept; however, some experimental results did not support Thomson's theory. A. A. Michelson and E. W. Morley hypothesized that the velocity of the ether with respect to the Earth would cause a slight, but measurable, shift in the speed of light. They designed an experiment using an interferometer placed in varying orientations to measure this shift. But, contrary to the physicists' expectations, the Michelson-Morley experiment found no shift in the speed of light. This contradicted the concept of the ether, therefore causing skepticism about Thomson's theory [4]. During the years that Tait and Thomson were developing the Theory of Vortex Atoms, another researcher was also looking for a theory on the structure of matter. Dimitri Mendeleev (1834-1907) based his theory on the assumed existence of atoms as well; he analyzed the arithmetic relationship between the elements atomic masses, and in 1869 put together a table based on similarities and patterns he observed between chemical properties of certain elements. Mendeleev published the Periodic Table of the Elements in 1872, which was widely accepted by the scientific community [19]. This resulted in Thomson's Theory of Vortex Atoms becoming almost immediately obsolete, and Knot Theory was virtually forgotten as well. Almost twenty years later, the theory of knots was reconsidered, by mathematicians this time, and Knot Theory took a very different direction.

III. Development of Knot Theory as a Viable Field of Study

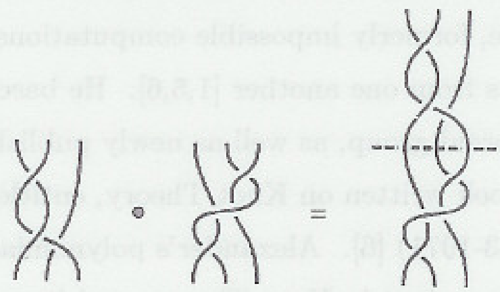
Braid Theory

In the 1920's, a mathematician was interested in applying Knot Theory to another area of study, but this time it was a purely mathematical concept. Braid Theory was developed by Emil Artin in the early 1920's. The applications of Braid Theory ranged from quantum mechanics to combinatorics to the textile industry, and soon Knot Theory would be added to this list as well. A braid is defined as "a set of *n* strings, all of which are attached to a horizontal bar at the top and at the bottom . . . [such] that each string intersects any horizontal plane between the two bars exactly once," [1]. Artin declared that the arrangement of the strands of a braid can be altered to achieve two representations of the same braid. He classified braids using the "braid word," a sequence of generators which could be used to construct the braid, as in the following figure. To construct the braid word of an n-stranded braid, begin by defining the two generators, σ_i and σ_i^{-1} as in the figure (a). The braid word is the list of generators necessary to construct the given braid, beginning from the top bar, and traveling down to the bottom bar. For example, the braid word for the figure (b) is $\sigma_2\sigma_3^{-1}\sigma_3\sigma_1^{-1}\sigma_1\sigma_2^{-1}$ [1,7,16].



(a) (b)

Artin defined the composition of two braids as the process of placing two braids end to end, as in the figure [1].



Using the definition of braid composition, Artin identified an n-stranded identity braid, I_n, as the braid that satisfies the relation $BI_n=B$. He proved the existence of a braid inverse, B^{-1} which has the property $B^{-1}B=I=BB^{-1}$, and proved associativity of braids, which ultimately revealed that an n-stranded braids form a group [1,16,18].

Of Artin's discoveries, possibly the most closely related to Knot Theory was Artin's comparison of braids by a method he called "combing." Artin published a theorem stating that "Two braids are isotopic if and only if the word representing one of them can be transformed into the word representing the other by a sequence of admissible calculations," [18]. These "admissible calculations" include, for example, eliminating both $\sigma_i\sigma_i^{-1}$ and $\sigma_i^{-1}\sigma_i$ from the given braid word [1]. Later, it will become clear that these operations are similar to the second Reidemeister move for a knot.

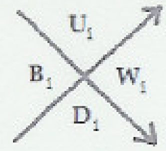
The Work of Alexander; The Alexander Polynomial

It is clear that if the ends of the n strands of a braid are connected resulting in an operation called "closure," a knot or a link could result. Artin defined the braid index as the fewest number of strands required to create a closed braid corresponding to the given knot. One mathematician interested in knot theory noticed a potential connection between knots and braids. James Waddell Alexander (1888-1971) wanted to know which knots and links can be obtained by the closure of a braid. In 1923, he proved that every link can be represented as a closed braid. A corollary of this result is that every knot can be represented as a closed braid. Alexander's theorem caused many mathematicians to ask another (still unanswered) question: Can the classification of braids be used to classify knots? [16]

Alexander discovered a polynomial knot invariant in 1928, which allowed him to perform valuable, formerly impossible computations, and to distinguish many non-isotopic knots from one another [1,5,6]. He based his polynomial on Artin's concept of the braid group, as well as newly published information contained in the first book written on Knot Theory, entitled <u>Knottentheorie</u>, by Kurt Reidemeister (1893-1971) [6]. Alexander's polynomial was the first discovered polynomial invariant in Knot Theory, and it remained the only polynomial invariant until the Jones polynomial was discovered in 1984.

Alexander began by defining the Alexander matrix of a given knot or link diagram. To obtain the Alexander matrix of an oriented diagram, label the universe of the link in the following way (a):

(a)



(b)

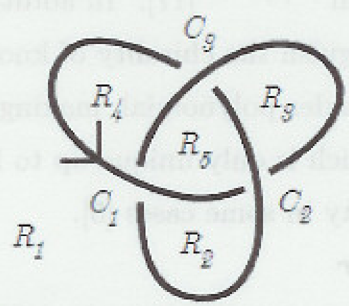
Notice that a knot with n crossings creates *(n+2)* regions in the plane of the diagram. Let R_k denote the kth region created by a given knot diagram, and let the crossing in diagram (b) above represent the labeling at the ith crossing. The Alexander matrix for a diagram with n crossings will be an n x *(n+2)* matrix with entries as follows:

$$A_{ij} = \begin{cases} 0 \text{ if } \mathrm{R}_j \text{ does not touch vertex } i \\ B_i, W_i, U_i, \text{ or } D_i \text{ if } \mathrm{R}_j \text{ touches vertex } i \text{ in that corner} \\ B_i + W_i, \text{ or } U_i + D_i \text{ if } \mathrm{R}_j \text{ touches 2 corners at vertex } i \end{cases}$$

in which B_i, W_i, U_i, and D_i correspond to the regions as labeled in the diagram (b) above. To find the reduced Alexander matrix, the last two rows are erased to form an n x n matrix. This reduced matrix, $\mathbf{A}_K(t)$, can then be used to calculate the Alexander polynomial of the knot. The Alexander polynomial is defined as

$$\Delta_K(t) \doteq \det(\mathbf{A}_K(t)),$$

where $\doteq$ represents equality up to factors of the form $\pm t^n$ [5]. For example, the Alexander polynomial of the trefoil knot can be computed in the following way: begin by labeling regions and crossings as in the figure



where C_i denotes the ith crossing, and R_i denotes the ith region. Given this labeling, the Alexander matrix associated with the trefoil knot is:

$$\begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \end{array}\begin{array}{c} \begin{array}{ccccc} R_1 & R_2 & R_3 & R_4 & R_5 \end{array} \\ \begin{bmatrix} -t & 1 & 0 & t & -1 \\ -t & t & 1 & 1 & -1 \\ -t & 0 & t & t & -1 \end{bmatrix} \end{array}.$$

The last two columns of the Alexander matrix are removed, and the reduced Alexander matrix becomes:

$$\mathbf{A}_{trefoil}(t) = \begin{pmatrix} -t & 1 & 0 \\ -t & t & 1 \\ -t & 0 & t \end{pmatrix}.$$

Then, to obtain the Alexander polynomial, compute the determinant of the reduced Alexander matrix:

$$\det(\mathbf{A}_{trefoil}(t)) = -t^3 + t^2 - t.$$

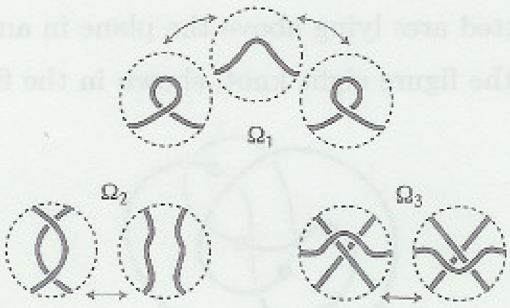
Since the Alexander polynomial is equal up to factors of the form $\pm t^n$, a t can be factored out of the determinant to obtain the final polynomial:

$$\Delta_{trefoil}(t) = -t^2 + t - 1.$$

Alexander's proof of his polynomial's invariance used algebraic topology, linear algebra, and Reidemeister moves [5,6]. Alexander's polynomial was a major discovery in Knot Theory, although it was not a complete invariant. There exist non-isotopic knots with equal Alexander polynomials. In particular, there are non-trivial knots with $\Delta(t) = 1$ [17]. In addition, the Alexander polynomial is unable to distinguish the chirality of knots. In the 1960's, John Conway normalized the Alexander polynomial, making it unique, (as opposed to the Alexander polynomial, which is only unique up to factors of the form $\pm t^n$), and able to distinguish chirality in some cases [5].

The Work of Reidemeister

Kurt Reidemeister became interested in Knot Theory in the 1920's, and his work on the subject pertained largely to planar diagrams of knots. Initially, Reidemeister struggled to create a new method of knot classification. He evaluated several techniques, including an endeavor to represent knots with equations. When neither an analytic nor a combinatorial approach provided adequate information to create a diagram of the knot or manipulate the knot, Reidemeister turned to the method of classification by diagram [16]. Reidemeister, using knot diagrams similar to those published by Tait, Little, and Kirkman, (in which over and under crossings are specified) proved that "Two knots K, K' with diagrams D, D' are equivalent if and only if their diagrams are related by a finite sequence $D=D_0, D_1, \ldots, D_n=D'$ of intermediate diagrams such that each differs from its predecessor by one of the following three ... Reidemeister moves," [15].

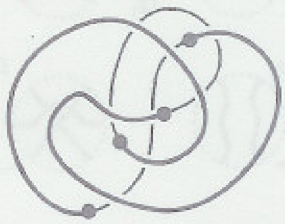


Reidemeister's theorem required a complicated proof, and provided knot theorists with an essential result that would assist in the establishment of knot invariants. Reidemeister began with the assumption that two knots, K and K' are equivalent, each having an equivalent diagram, D and D' respectively. Knowing that one can obtain K' from K using ambient isotopy, it follows that one can obtain D' from D using a finite number of operations. Reidemeister showed that one is able to divide each diagram into parts such that each part contained either a single arc or a single crossing. From this point, he analyzed all of the possible operations which could be performed on these diagram fragments, and ultimately proved that the only possibilities were the three Reidemeister moves [15].

Reidemeister showed that three operations were sufficient to represent ambient isotopy, although it was demonstrated later that in certain instances the first Reidemeister move is actually not necessary. (The relation defined by using only the second and third Reidemeister moves is referred to as "Regular Isotopy") [6,16]. Interestingly, the moves that Reidemeister used had been defined several years previously by Maxwell. The significant element of Reidemeister's work was not the proof that one could transform one knot into another equivalent knot using Reidemeister moves, but the proof that these three moves were the *only* three needed to illustrate the equivalence of two knots. This would become a vital factor in the impending development of certain knot invariants.

The Work of Schubert

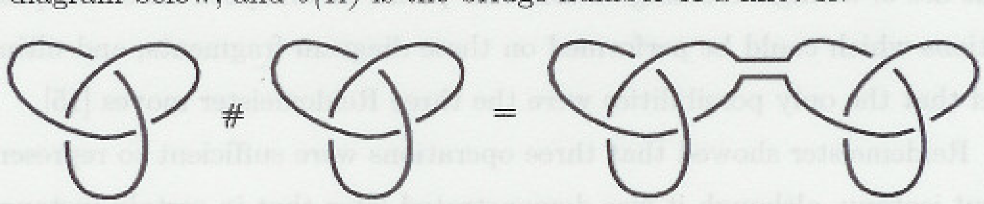
In the 1940's, mathematician Horst Schubert also approached knots from an arithmetic perspective. His studies produced several important findings, one of which was the invariant called the "bridge number," which is defined as "The least number of unknotted arcs lying above the plane in any projection," [18]. The bridge number of the figure eight knot, shown in the figure below, is two [1].



Note that the only knot associated with bridge number one is the unknot. Schubert defined the bridge number, and showed its additivity. Schubert's most critical contribution to Knot Theory was his proof, using the bridge number, that every knot can be uniquely decomposed into prime knots [16]. Schubert proved in 1954 that

$$b(K_1 \# K_2) = b(K_1) + b(K_2) - 1,$$

where $K_1 \# K_2$ represents the connected sum of two knots K_1 and K_2, as illustrated in the diagram below, and $b(K)$ is the bridge number of a knot K.



The knots which have bridge number two, such as the figure eight knot, are known as *two-bridge knots*. Schubert used the bridge number of the connected sum to show that all two-bridge numbers are prime.

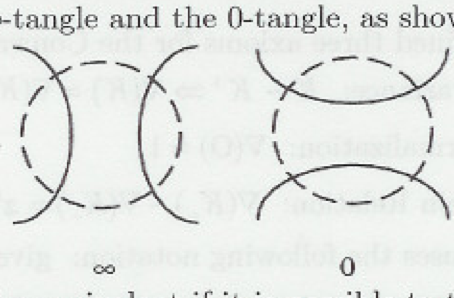
Knot Theory was beginning to attract significant attention and would continue to be at the center of a growing body of research.

IV. Discovery of Knot Invariants

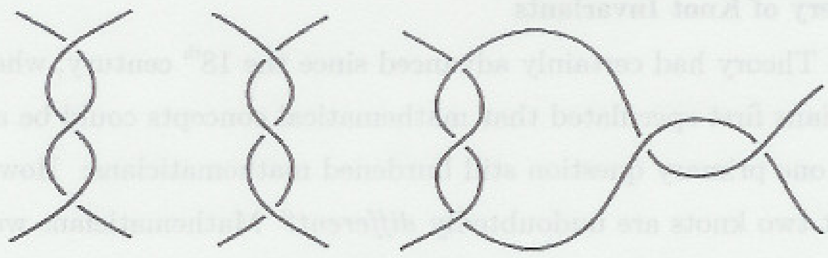
Knot Theory had certainly advanced since the 18th century, when mathematicians first speculated that mathematical concepts could be applied to knots. But one primary question still burdened mathematicians: How can one confirm that two knots are undoubtedly *different?* Mathematicians wanted to find knot invariants which would remain unchanged under ambient isotopy. These invariants would allow one to identify two equivalent knots given isotopically different diagrams. The first known knot invariants are the aforementioned Linking Number, discovered by Gauss, the Alexander Polynomial, and Schubert's bridge number. Reidemeister's proof that the Reidemeister moves were sufficient to transform one knot into another if the two are equivalent made the forthcoming invariants easier to identify.

The Work of Conway; The Conway Polynomial

In the 1960's, English mathematician John Conway (1937-) developed a new method for knot notation, and during his study of knots, rediscovered and normalized the Alexander polynomial. Conway's knot notation begins with his definition of a "tangle," identified by W. B. R. Lickorish. A tangle can be defined as "A region in a knot or link projection plane surrounded by a circle such that the knot or link crosses the circle exactly four times." [18]. The crossings occur in the four compass directions: NE, NW, SE, SW. The most basic examples are the ∞-tangle and the 0-tangle, as shown in the figure.



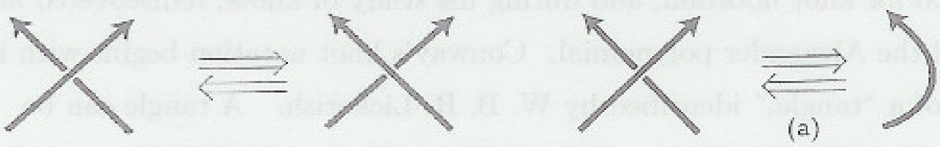
Two tangles are said to be equivalent if it is possible to transform one into another using a series of Reidemeister moves while keeping the endpoints fixed. A tangle consisting of $\pm n$ twists, (which can be thought of as the tangle obtained by applying the first Reidemeister move $\pm n$ times, resulting in $\pm n$ crossings), is called a $\pm n$-tangle, with the sign used to indicate the direction of the twists, as in the examples below [1]. Conway's knot notation is obtained by connecting the ends of appropriate tangles to form the specified knot, then listing the tangles.



(a) (b) (c)

Figures (a) and (b) show 3-tangles, and figure (c) shows a tangle consisting of a 3-tangle and a 2-tangle, denoted "3, 2." Conway's knot notation became widely accepted, as it provides a simpler method of documentation for knots up to eleven crossings [1,17].

Although Alexander had employed these operations years before, Conway defined a "flip" as the process of "transforming the chosen crossing (on the planar representation of the knot) into the opposite crossing," and "smoothing" as the process of resolving the chosen crossing, as in the figure.

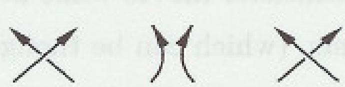


(flip) (smoothing)

Conway was searching for a way to be certain that two inequivalent knots are different, and what he found was actually a modified version of the Alexander polynomial. Conway defined three axioms for the Conway polynomial, $\nabla(x)$:

1. Invariance: $K \sim K' \Rightarrow \nabla(K) = \nabla(K')$.
2. Normalization: $\nabla(\mathrm{O}) = 1$.
3. Skein Relation: $\nabla(K_+) - \nabla(K_-) = x\nabla(K_0)$.

Conway's skein relation uses the following notation: given three knot diagrams which are identical with the exception of one crossing, denote the diagrams as follows



Conway's polynomial is related to Alexander's polynomial as follows:

$$\Delta_K(t) \doteq \nabla_K\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right).$$

Conway used Reidemeister moves to prove the invariance of his polynomial. However, Conway's polynomial is not a complete invariant of knots. There are non-isotopic knots having the same Conway polynomial. However, the Conway polynomial can distinguish the chirality of a knot in some cases, which the Alexander polynomial cannot [6]. Since Conway's polynomial is incomplete, knot theorists were still in search of a more sensitive polynomial.

The Work of Jones; The Jones Polynomial

In May of 1984, Vaughan Jones (1952 -) found a correspondence to Braid Theory during his study of von Neumann algebras. He wrote, "'In my work on von Neumann algebras, I had been astonished to discover expressions that bore a strong resemblance to the algebraic expression of certain topological relations among braids. I was hoping that the techniques I had been using would prove valuable in knot theory. Maybe I could even deduce some new facts about the Alexander polynomial,'" [13].

The Jones polynomial is a polynomial in t which satisfies the following three axioms:

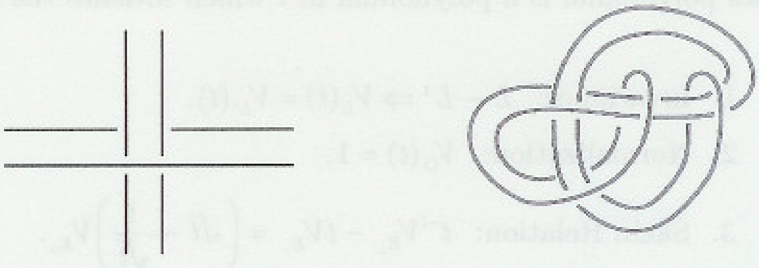
1. Invariance: $L \sim L' \Rightarrow V_L(t) = V_{L'}(t)$.
2. Normalization: $V_O(t) = 1$.
3. Skein Relation: $t^{-1}V_{K_+} - tV_{K_-} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)V_{K_0}$.

Jones proved the invariance of his polynomial under ambient isotopy; the Jones polynomial could now be used to distinguish two inequivalent knots from one another. This was the only polynomial invariant discovered since the Alexander polynomial, and it would prove to be very powerful.

The Jones polynomial proved to be a more sensitive invariant than that of Alexander, even after Conway's normalization. After its discovery, the Jones polynomial was calculated for knots up to thirteen crossings. All of these knots were associated with unique polynomials, with the exception of two knots, each with eleven crossings. Meticulous examination of these two knots revealed that they were equivalent, and the knot table was corrected [16].

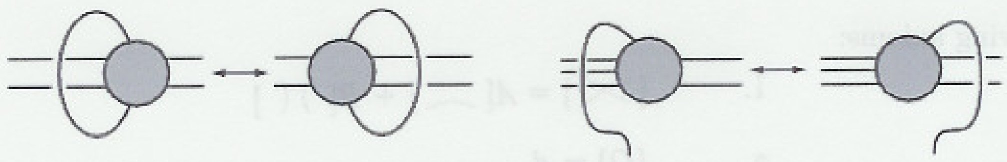
The discovery of the Jones polynomial was critical to Knot Theory. It was the first polynomial invariant able to distinguish a knot's handedness. In particular, it distinguishes the right trefoil from the left trefoil, a discrimination which was impossible with the Alexander or Conway polynomials. As mentioned in Section II, the Jones polynomial was used in the first rigorous proof of the Tait Conjectures. While the Jones polynomial is a sensitive invariant, it is not a complete invariant. There exist nonisotopic knots which have the same Jones polynomial. Since its discovery, the Jones polynomial has been derived using several different methods, and has been found in fields seemingly unrelated to Knot Theory, such as statistical mechanics, which will be mentioned in Section V.

The Jones polynomial can be simplified to obtain an invariant called the Arf invariant, which always has a value of zero or one. In 1965, Raymond Robertello was studying *ribbon knots,* which can be defined as knots which bound a disk containing self intersections of the type illustrated in the figure (a) below [8,18]. Figure (b) is an example of a ribbon knot.



(a) (b)

Robertello used pass equivalence classes to distinguish ribbon knots from those which are not ribbon. Given an operation known as a "pass move," which can be defined as "A change in a knot projection such that a pair of oppositely oriented strands are passed through another pair of oppositely oriented strands," [18] Robertello found that all knots fall into two distinct pass equivalence classes: the pass equivalence class of the unknot, or that of the trefoil.



(Examples of pass moves)

Since all ribbon knots fall into the same pass equivalence class as the unknot, he proved that those knots which are pass equivalent to the trefoil are not ribbon. Using the fact that every knot bounds an orientable surface, or Seifert surface, Robertello found that the standard Arf invariant of quadratic forms, when computed on the quadratic form associated to a Seifert surface of a knot, distinguishes the pass equivalence class of a knot. The Arf invariant in Knot Theory can be defined as the second degree coefficient of the Conway polynomial modulo two [6]. It can also be expressed as the value of the Jones polynomial at $t = i$:

$$\mathrm{Arf}(K) = V_K(i)$$
$$i = \sqrt{-1}.$$

Four months after the discovery of the Jones polynomial, an invariant called the HOMFLY polynomial was discovered by six mathematicians - Hoste, Ocneanu, Millett, Freyd, Lickorish, and Yetter, whose names constitute the acronym HOMFLY. They discovered the polynomial invariant independently and published their findings in the same publication. The HOMFLY polynomial is a polynomial in a and z which satisfies the following three axioms:

1. Invariance: $L \sim L' \Rightarrow P_L = P_{L'}$.
2. Normalization: $P_O = 1$.
3. Skein Relation: $a^{-1}P_{L+}(a,z) - aP_{L-}(a,z) = zP_{L0}(a,z)$.

The HOMFLY polynomial is a generalization of the Jones polynomial, and in most cases detects chirality [1,16].

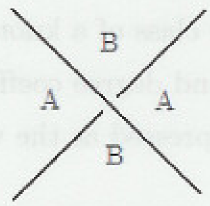
The Work of Kauffman; The Bracket Polynomial

In August 1985, mathematician Louis H. Kauffman (1945 -), employing techniques used in the study of statistical physics, discovered another approach to the Jones polynomial. Two weeks before this finding, he had discovered a generalized form of another polynomial called the BLM/Ho invariant [16]. He began by defining a polynomial in variables A, B and d which satisfied the following axioms:

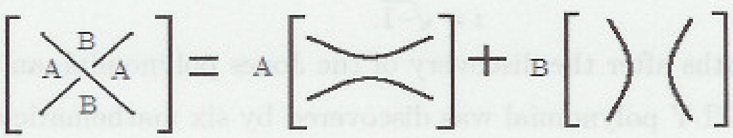
1. $[\times] = A[\asymp] + B[\,)(\,]$

2. $[O] = d.$

In the second axiom, O represents the unknot. Using the observation that each crossing on a diagram creates two pairs of complimentary angles on the plane, as in the figure below, Kauffman defined region A as the area on one's right if walking along the upper strand toward the crossing, and region B as the area on the left.



Using this convention, the first formula above becomes:

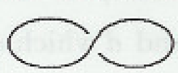
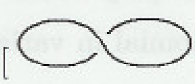


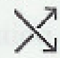
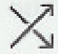
Kauffman's formula states that each crossing in an unoriented diagram can be smoothed in two ways: either by an A-splicing connecting the A regions, or by a B-splicing connecting the B regions [6].

Kauffman defined the *state, S,* of a given universe, *U,* of a knot as "a choice of splitting for each vertex of *U*," [6]. There are 2^n possible states associated with a universe with n vertices. Kauffman's "Bracket polynomial" could then be defined by the following "state sum" formula:

$$[K] = \sum_s A^{\alpha(s)-\beta(s)} d^{\gamma(s)-1},$$

where $\alpha(s)$ is the number of A-splicings, $\beta(s)$ is the number of B-splicing, and $\gamma(s)$ is the number of components of the state [1,6,16]. It is easily verified that given the conditions $B=A^{-1}$ and $d=(-A^2-A^{-2})$, the Kauffman Bracket is invariant under the second and third Reidemeister moves. However, the first Reidemeister move presented a problem. For example:

[] $= -A^3 \neq$ [] $= -A^{-3}$.

The Kauffman Bracket is not invariant under the first Reidemeister move. Kauffman searched for a modification to his polynomial that would result in its invariance under ambient isotopy. Recall that the writhe, $w(\mathrm{K})$, of a knot or link is defined as the sum of all the crossing signs on a diagram, given that  corresponds to +1 and  corresponds to -1. The writhe is invariant under the second and third Reidemeister moves, but not under the first, which changes the value of $w(\mathrm{K})$ by plus or minus one. Kauffman employed this fact and showed that

$$V_K = (-A)^{-3w(K)}\,[K].$$

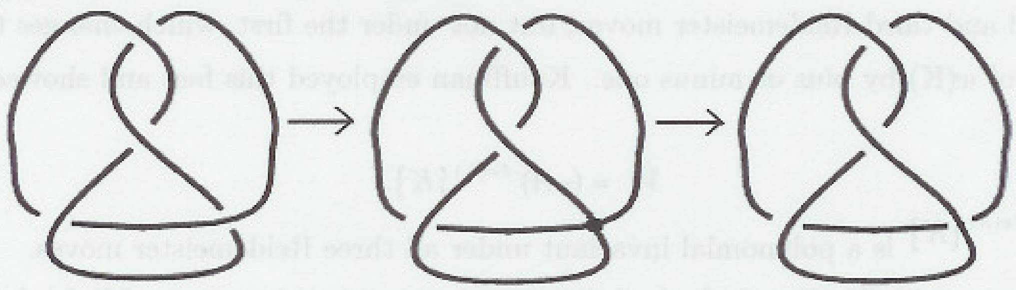
$(-A)^{-3w(K)}\,[K]$ is a polynomial invariant under all three Reidemeister moves. Kauffman initially thought he had discovered an original invariant of links, but soon he realized that he had discovered a different method of obtaining the Jones Polynomial [6,16].

Although the Kauffman Bracket polynomial only produced an alternative method for computing the Jones polynomial, it is certainly relevant to Knot Theory, and ideas introduced in the discovery of the polynomial were applicable to several disciplines outside of Knot Theory. The Bracket polynomial has been used in the study of physics and statistics, both of which will be addressed in Section V.

The Work of Vassiliev; Vassiliev Invariants

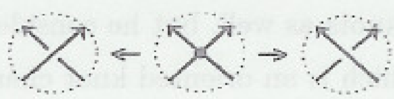
Victor Vassiliev took a different approach to the study of knots altogether, perhaps because Knot Theory was not his primary area of study. Vassiliev studied singularities, in the framework of the so-called "catastrophe theory," and he applied his knowledge in this area to Knot Theory. Until this point, the only knots studied had been those with over and under crossings, or "ordinary knots." Vassiliev studied ordinary knots as well, but he considered another type of knot, called a "singular knot," which is an oriented knot characterized by having what he called a "double point," [16].

To understand the concept of a double point, one can think of a knot moving in space, with the arcs able to cross through each other. Two arcs could approach each other, then cross through one another, and the crossing would be reversed. However, there is an instant, a "catastrophic moment," during this crossing switch in which the arcs would be in the same plane. Neither one would be crossing over or under the other; the arcs would meet at a point, called a double point, which is represented by a dot in the following figure. In the next instant the knot would again be an ordinary knot and the crossing would be the opposite of what it was before the reversal.



Vassiliev defined the set of all knots, $\Im$, an infinite set containing ordinary knots and singular knots with any finite number of double crossings. He then defined certain subsets of $\Im$; for example, $\Sigma_0 \subseteq \Im$ is defined as the set of all ordinary knots; $\Sigma_1 \subseteq \Im$ is the set of singular knots containing exactly one double point, etc. He used these subsets in his search for knot invariants. Vassiliev discovered that by studying each subset Σ_i of $\Im$, one is able to define a vector space over the real numbers, V_i, of all of the real valued invariants that vanish on knots with more than i double points [16,18].

The notion of ambient isotopy for singular knots is similar to that of ordinary knots, with one significant distinction: ambient isotopy for ordinary knots cannot instantaneously change an over crossing to an under crossing, or vice versa; if one considers a knot with a chosen double point, using ambient isotopy, one is able to perturb one of the chosen arcs to obtain either an over crossing or an under crossing. Specifically,



This idea led Vassiliev to an interesting observation regarding his invariants. He defined the order of a Vassiliev invariant in the following way: a Vassiliev invariant is of order n if it vanishes on knots with more than n double points

[16]. He then proved the following:

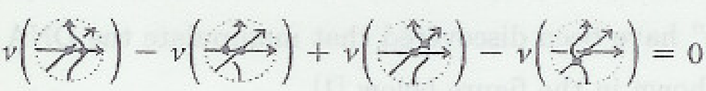
Lemma: *"The value of the Vassiliev invariant of order less than or equal to n of a singular knot with exactly n double points does not vary when one (or several) crossings are changed to opposite crossings,"* [16].

Since it can be shown that there exist no nontrivial first order invariants, it follows from the Lemma that all zero order and first order invariants are equal to that of the unknot. The values vary for orders greater than or equal to two.

Vassiliev invariants, like the invariants discussed earlier in this section, follow a set of axioms. Firstly, if two knots have the same number of double points, their Vassiliev invariants are equal. Secondly, every Vassiliev invariant, regardless of its order, satisfies the following skein relation:

in which the dot represents a double point. Following this relation, one is able to obtain the one-term relation:

and the four-term relation:



The discovery of Vassiliev invariants was fundamental to the study of knots. Not only did Vassiliev discover new knot invariants, he discovered an entirely new method of searching for knot invariants. While the previously known polynomial invariants are not complete, there is a conjecture (which is currently unproved) stating that "for each pair of nonequivalent knots K_1 and K_2 there is a natural number $n \in N$ and an invariant $v \in V_n$ such that $v(K_1) \neq v(K_2)$," [16]. Currently, even with Vassiliev's important discovery, there remain undiscovered invariants. Vassiliev's approach, however, allows mathematicians to define new invariants for knots. It is thought that Vassiliev's method can be applied to other disciplines, such as physics, which will be discussed in the following section [16].

V. Applications of Knot Theory to Other Disciplines

Since the original Knot Theory was applied to Thomson's Theory of Vortex Atoms, several connections have been drawn between the study of knots and certain areas in Physics and Biology. These connections have arisen from the observation of a number of "coincidences," or similarities between formulae appearing in seemingly disconnected fields. As the examination of these presumed coincidences continues, several new discoveries have been made, both in the form of knot invariants and alternative approaches to already known invariants. In many cases, concrete connections have yet to be made, but the analysis of these similarities continues with the hope that the application of knots to other disciplines will promote the discovery of new methods, new invariants, and previously unknown properties of knots [16].

While the first practical applications of Knot Theory were to physics, connections between the study of knots and biology have also been discovered. In the 1950's, it was found that DNA molecules take the shape of a double-helix. DNA molecules perform several biological functions, some of which can be affected by the topological properties of the DNA strands. Enzymes called "topoisomerases" have been discovered that manipulate the DNA strands using the operations shown in the figure below [1].

Biochemists identified the effects of these topoisomerases on DNA by observing their effects on closed, or circular, DNA molecules, then describing the resulting knotted DNA using properties of knots such as the writhe and linking number [1]. The study of knots as applied to the topology of DNA shows great promise in furthering our understanding of biochemistry.

A profound example of the ties between physics and Knot Theory is the discovery of actual statistical models which provide an alternate approach to the calculation of knot invariants. Recall that Kauffman discovered the Bracket polynomial using a technique associated with statistical models. Kauffman's polynomial was not itself a statistical model for any existent object. However, following the introduction of the idea that statistical models could be related to polynomial invariants, Jones discovered a real statistical model that could be used to derive certain knot invariants [16].

Jones began with two actual statistical models, the Ising model and the Potts model. The Ising model was developed in 1924 by Ernst Ising during his study of ferromagnetic materials. It is a statistical model in which particles in the system adapt their behavior according to the behavior of the particles in their immediate surroundings. Each statistical model is associated with a partition function, or the sum of terms over all of the possible states of the system. Jones found that the partition function associated with the Ising model can be used to derive the Arf invariant [1,16].

Jones also studied a generalized form of the Ising model, called the Potts model, to obtain an alternate derivation of the Jones polynomial. The Potts model is used to model the phase change from water to ice. Jones analyzed the partition function associated with the Potts model; using the relation

$$|S| = 2 + t + \frac{1}{t},$$

where $|S|$ denotes the number of states in the system, and t is the variable used in the Jones polynomial, Jones discovered that the partition function gives the Jones polynomial. While there is much still undiscovered on the subject of knots related to statistical physics, Jones' discovery has motivated mathematicians to search for new knot invariants by means of statistical models [16].

Another seemingly promising link between physics and knots pertains to Vassiliev invariants. Vassiliev invariants were discovered during the early 1990's, making them the youngest of the previously discussed invariants. Distinct connections between the invariants and physics have yet to be made, but there is reason to expect that correlations will be found as research progresses [16]. It is clear that the notion that Knot Theory can be related to other fields is a work in progress.

VI. Conclusion

Knot Theory has progressed considerably from its early rise to prominence as Thomson's failed idea of the knot as a model for the atom. However, the central questions today are still as elusive as they were then: How can we classify knots effectively; can we establish a complete and efficient system of invariants; and what role do knots play in the basic theory of the structure of matter? These challenges ensure that Knot Theory will continue to be a dynamic field of study for the scientific world.

References

[1] C. C. Adams, *The Knot Book, W. H. Freeman and Company,* (2001).

[2] J. Baez, *Gauge Fields, Knots and Gravity, World Scientific,* 4 (1994), 310-327.

[3] Famous Scientists of Thermodynamics, http://about-thermodynamics.com.

[4] D. C. Giancoli, *Physics for Scientists & Engineers, Prentice Hall,* 3 (2000), 709-726, 919-922.

[5] L. H. Kauffman, *Formal Knot Theory, Princeton University Press* 30, (1983), 95-164.

[6] L. H. Kauffman, New Invariants in the Theory of Knots, *The American Mathematical Monthly* 110, (2003), 195-242.

[7] K. L. Kristiansen, Some Applications of Knot and Braid Theory in Experimental Physics, *University of Oslo*, (2000).

[8] C. Livingston, *Knot Theory, The Mathematical Association of America* 24, (1993), 1-141.

[9] H. Miller, Knots and Numbers, *MIT University,* (2003).

[10] C. Nash, Topology and Physics - a historical essay, *National University of Ireland,* (1997).

[11] J. J. O'Connor, Topology and Scottish mathematical physics, *University of St. Andrews,* (2001).

[12] J. H. Przytycki, History of Knot Theory, http://www.maths.ed.ac.uk/~aar/knots.

[13] J. H. Przytycki, Knot Theory from Vandermonde to Jones, *Aportaciones Matematicas Comunicaciones* 11 (1992), 173-185.

[14] R.L. Ricca, New Developments in Topological Fluid Mechanics: From Kelvin's Vortex Knots to Magnetic Knots, *Ideal Knots, World Scientific* 19 (1998), 255-273.

[15] J. Roberts, Knots Knotes, *Edinburgh University*, (1999), 1-67.

[16] A. Sossinsky, *Knots: Mathematics with a Twist, Harvard University Press,* (2002), 1-119.

[17] M. Thistlewaite, Aspects of Topology, *London Mathematical Society* 93, (1985), 1-76.

[18] E. W. Weisstein, Mathworld -- A Wolfram Web Resource. http://mathworld.wolfram.com

[19] S.S. Zumdahl, *Chemical Principles, Houghton Mifflin Company* 5, (2005) 548-549.