

GAUSS MAP AND THE SECOND FUNDAMENTAL FORM

Let S be a regular orientable surface, and $\varphi : U \rightarrow S$ be a parametrization of S . Let $\{e_{u_1}, e_{u_2}\}$ be the corresponding basis of the tangent plane $T_p S$ to the surface at a point $p = \varphi(u_1, u_2)$. Let $N(p) = \frac{e_{u_1} \wedge e_{u_2}}{|e_{u_1} \wedge e_{u_2}|}$ be the unit normal vector to S , compatible with the chosen orientation. Note that together the three vectors (e_{u_1}, e_{u_2}, N) form a basis of \mathbb{R}^3 , with the center at p .

The *Gauss map* is the map $N : S \rightarrow S^2$ defined by the normal vector. That is, for each point of the surface $p \in S$ the result of the map N is the unit normal vector $N(p) \in S^2$. Note that the normal vector to S at p is parallel to the normal vector to S^2 at $N(p)$. Thus, $T_p S$ is parallel to $T_{N(p)} S^2$. Therefore, the differential of the Gauss map, $dN_p : T_p S \rightarrow T_{N(p)} S^2$, can be considered as an operator on the tangent plane $T_p S$. This operator turns out to be self-adjoint, and thus, (its negative) defines a symmetric bilinear form on $T_p S$. This form is called the second fundamental form. We have

$$II_p(w) = - \langle dN_p(w), w \rangle \quad \forall w \in T_p S,$$

where brackets denote the standard scalar product.

The coefficients of the second fundamental form are denoted by $e = - \langle N_{u_1}, e_{u_1} \rangle$, $f = - \langle N_{u_1}, e_{u_2} \rangle = - \langle N_{u_2}, e_{u_1} \rangle$ and $g = - \langle N_{u_2}, e_{u_2} \rangle$. In practical computations of the coefficients, it is convenient to use the identity $\langle N, e_{u_i} \rangle = 0$ (which holds simply because $N \perp T_p S$). This allows to obtain the following formulas:

$$\begin{aligned} e &= \langle N, e_{u_1 u_1} \rangle \\ f &= \langle N, e_{u_1 u_2} \rangle \\ g &= \langle N, e_{u_2 u_2} \rangle \end{aligned}$$

Exercise 1. Choose your favorite parametrized surface and compute the matrix of its second fundamental form with respect to your favorite parametrization. (Hint: compute the basis corresponding to the parametrization, the normal vector, the derivatives of the basis vectors, and then apply the formulas above).

Gaussian curvature at a point $p \in S$ is given by $K = (\det(dN_p))$ and *mean curvature* at p is $H = -\frac{1}{2} \text{tr}(dN_p)$. Note that it is very convenient to have this definition in terms of the determinant and the trace of a matrix of dN_p for the following reason: since matrices of the same operator with respect to different bases are similar, and the determinant and the trace of two similar matrices are the same, this definition allows to compute K and H once we know dN_p in any basis (e.g., in the basis associated to a given parametrization). There is a classification of points (elliptic, hyperbolic, parabolic or planar points) depending on whether the Gaussian curvature is positive, negative, or has one or both eigenvalues equal to 0 respectively. (See p. 146 of the book for more details).

The eigenvalues of dN_p are called the *principal curvatures* of S at p and denoted by k_1 and k_2 . In terms of these curvatures, $K = k_1 k_2$ and $H = \frac{1}{2}(k_1 + k_2)$.

For a unit-speed curve $\gamma(s) \in S$, let $p = \gamma(s_0)$ be a point on the curve, and $w = \gamma'(s_0)$ be the tangent vector to the curve at this point. The “acceleration” vector $\gamma''(s_0)$ has two components, one, denoted by γ''_T , belongs to the tangent plane $T_p S$, and the other, denoted by γ''_N , is parallel to the normal vector N at p , so that

$$\gamma'' = \gamma''_T + \gamma''_N.$$

Recall that $|\gamma''| = k$, the usual curvature of the curve γ at the given point. The length $|\gamma''_T|$ is called the *geodesic curvature* at p and is denoted by k_T . The length $|\gamma''_N|$ is called the *normal curvature* at p and is denoted by k_N . One can show that $k_N(\gamma'(s_0)) = II_p(\gamma'(s_0))$. Thus, the normal curvature of a curve at a point p depends only on the tangent vector to the curve at this point. This allows on to speak of a *normal curvature* $k_N(w)$ in the given direction of a vector $w \in T_p S$.

Note that the relation $II_p(w) = k_N(w)$ holds only for unit vectors. For a vector of an arbitrary (non-zero) length, $k_N(w) = II_p(w/|w|) = II_p\left(\frac{w}{\sqrt{I_p(w)}}.$

Exercise 2. Fix $h \in (-1, 1)$ and let

$$\gamma(\theta) = (\sqrt{1-h^2} \cos \theta, \sqrt{1-h^2} \sin \theta, h),$$

where $\theta \in [0, 2\pi]$, be a “horizontal” circle on unit the sphere, at height h . Compute the normal curvature of $\gamma(\theta)$ (by symmetry, it is the same at all points of the curve). Compute the geodesic curvature. Note how these curvatures behave as functions of h . What are the values of h that correspond to the zero normal curvature? (Hint: first, check whether $\gamma(\theta)$ is a unit speed curve). Draw several pictures for various values of h , indicating on each picture the vectors $\gamma''(s_0)$, $\gamma''_T(s_0)$ and $\gamma''_N(s_0)$ and the normal vector N at a point $\gamma(s_0)$ on the curve.

Since the principal curvatures k_1, k_2 are the eigenvalues of a self-adjoint linear operator dN_p on $T_p S$, there is a corresponding orthonormal basis (e_1, e_2) of eigenvectors of dN_p . The directions of these two vectors are called the *principal directions* of S . Let w be a unit vector in $T_p S$. Since $w = \cos \theta \cdot e_1 + \sin \theta \cdot e_2$ for some θ , one can compute the normal curvature in the direction of w using the principal curvatures:

$$k_N(w) = II_p(w) = \cos^2 \theta \cdot k_1 + \sin^2 \theta \cdot k_2$$

This, in particular, shows that normal curvature in any direction satisfies

$$k_N \in (\min(k_1, k_2), \max(k_1, k_2))$$

(Note that if a given vector w is not unit, it should be replaced by $w/|w|$ in the computation of k_N using the second fundamental form). Moreover, every value of the normal curvature between $\min(k_1, k_2)$ and $\max(k_1, k_2)$ is achieved by some direction. For example, if both principal curvatures are positive (i.e., the point is elliptic), then all normal curvatures are positive. On the other hand, at a hyperbolic point, where the principal curvatures have opposite signs, there are some directions (called the *asymptotic directions*) in which the normal curvature is 0.

Exercise 3. Let $p \in S$ be a point on a regular orientable surface, such that the principal curvatures at this point are $k_1 = 1$ and $k_2 = 4$. Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$ be the

corresponding principal directions. Find all the unit vectors $w \in T_p S$ in the direction of which the principal curvature is equal to $2\frac{1}{2}$.

As we already saw above, the operator dN_p plays a major role in the study of local behavior of a surface. So, it is very important to be able to compute the value of this operator on a given vector in the tangent plane, as well as its matrix with respect to the basis corresponding to a given parametrization. The first of these computations can be done in the following way: given a vector $w \in T_p S$, represent w as a tangent vector to a curve, $w = \alpha'(0)$ for some curve $\alpha : (-\varepsilon, \varepsilon) \rightarrow S$, with $\alpha(0) = p$. Let $N(t)$ be the restriction of the normal vector to this curve. Then $dN_p(w) = N'(0)$. For an example of a such computation, see examples 2-4 on pages 137-140.

Given a basis (e_{u_1}, e_{u_2}) of $T_p S$ corresponding to a given parametrization, the matrix A of the operator dN_p can be computed using the first and second fundamental forms, as follows:

$$A = -[II_p] \cdot [I_p]^{-1},$$

where $[I_p]$ and $[II_p]$ denote matrices of the fundamental forms with respect to the same basis. In particular, it follows that $K = \frac{\det[II_p]}{\det[I_p]}$.

Exercise 4. For the surface and parametrization you chose in Exercise 1, compute the matrix of the operator dN_p , the Gaussian and the mean curvature.