Borel reducibility and classifying factors

R. Sasyk (Bs. As.)
joint work with A. Törnquist (Vienna)

Univ. Nac. General Sarmiento
Buenos Aires, Argentina
rsasyk@ungs.edu.ar

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Without a requirement on f, the definition would only amount to studying the cardinality of X/E vs. Y/F.

Smooth equivalence relations

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Example 2 (Ornstein-Bowen): X Classical Bernoulli shifts, E conjugacy. $f(T) =$ the entropy of T.

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Theorem (Baer)

The isomorphism relation for countable rank 1 torsion free abelian groups is Borel bireducible to E_0.

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Theorem (Woods '71): $E_0 \leq_B \mathrm{ITPFI}_{\simeq}$.

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3. $E_0 <_B E <_B E_\infty$ (Jackson, K, L)

Classification by countable structures

Definition. Let E be an equivalence relation on a Polish space X. E is *classifiable by countable structures* if

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Where S_∞ is the infinite symmetric group and $E^Y_{S_\infty}$ denotes a Borel equivalence relation induced by a continuous S_∞-action on Y.

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Example I: $E_\infty \leq_B E^Y_{S_\infty}$

Example II: (Halmos-vN) $E =$ conjugacy of ergodic m.p. transformations with discrete spectrum. $\sigma_P(T)$ is a complete invariant. $E \leq_B E^Y_{S_\infty}$

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Example V: (Epstein-Ioana-Kechris-Tsankov, '08) OE of G actions for G non amenable is not classifiable by countable structures.

Results: Theorem 1

Theorem (S.-Törnquist, '08)

The isomorphism relation for separable von Neumann factors of type II_1, II_∞ *and* III_λ, $\lambda \in [0,1]$, *are not classifiable by countable structures.*

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Corollary

The classification problem of II_1 *factors is not smooth.*

A factor $M \in \mathrm{vN}(H)$ is *injective* (or *amenable* or *hyperfinite*) if it contains an increasing sequence of finite dimensional von Neumann algebras, with dense union in M. For each of the types $\mathrm{II}_1, \mathrm{II}_\infty$ and III_λ, $\lambda \in (0, 1]$, there is a unique injective factor of that type. However, for type III_0 we have:

Results: Theorem 2

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The isomorphism relation for injective factors of type III_0 is not classifiable by countable structures.

(Compare with Woods' Theorem: $E_0 \leq_B \mathrm{ITPFI}_{2_{\simeq}}$.)

Results: Theorem 3

Denote by $\mathcal{F}_{\mathrm{II}_1}(\mathcal{H})$ the (standard) space of II_1 factors on $\mathcal{H}$, and by $\simeq^{\mathcal{F}_{\mathrm{II}_1}(\mathcal{H})}$ the isomorphism relation for factors of type II_1 on $\mathcal{H}$.

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As an immediate corollary, we have:

Corollary

The isomorphism relation for factors of type II_1 is complete analytic as a subset of $\mathcal{F}_{\mathrm{II}_1}(\mathcal{H}) \times \mathcal{F}_{\mathrm{II}_1}(\mathcal{H})$. In particular it is not a Borel subset.

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- Apply Popa's *deformation-rigidity* techniques ($\mathcal{HT}$ factors) to argue that the properties of the $\mathbb{F}_3$-actions carry over to properties of the corresponding factors,
- Argue that the family of $\mathbb{F}_3$-actions is too big to be classified by countable structures.

A class of $\mathbb{F}_3$-actions

The $\mathbb{F}_2$ action σ obtained by restricting the $SL_2(\mathbb{Z})$ action on $\mathbb{T}^2$ is free, ergodic, and $L^\infty(\mathbb{T}^2) \subset L^\infty(\mathbb{T}^2) \rtimes_\sigma \mathbb{F}_2$ has the relative property (T).

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$$\text{Ext}(\sigma) = \{S \in \text{Aut}(\mathbb{T}^2) : S, T_a, T_b \text{ generate a free } F_3\text{-action}\}.$$

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1. *(Törnquist) The relation $\sim_{oe}$ has meagre classes and all classes are dense.*

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1. *(Törnquist) The relation $\sim_{oe}$ has meagre classes and all classes are dense.*
2. *(Kechris-Törnquist) The relation $\sim_{oe}$ is generically turbulent, so in particular, it is not classifiable by countable structures.*

Proof of Theorem 1

We can now finish the proof of Theorem 1. For each $S \in \mathrm{Ext}(\sigma)$, let

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The map $S \mapsto M_S$ may be seen to be Borel. We claim it is a Borel reduction of $\sim_{oe}$ to isomorphism of von Neumann factors on $L^2(\mathbb{T}^2 \times \mathbb{F}_3)$. It is clear by Feldman-Moore's Theorem that if $S \sim_{oe} S'$ then $M_S \simeq M_{S'}$. For the converse, we invoke a deformation-rigidity result of Popa:

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Theorem (Popa, '01)

Suppose G is a countable group acting in a measure preserving a.e. free ergodic way on (X, μ) Then if $L^\infty(X)$ has both the relative property (T) and the relative Haagerup property as a subalgebra of $L^\infty(X) \rtimes G$, then $L^\infty(X)$ is, up to conjugation with a unitary, the only Cartan subalgebra of $L^\infty(X) \rtimes G$ with both the relative property (T) and the relative Haagerup property.

Proof of Theorem 1

Suppose then that $M_S \simeq M_{S'}$. Since $\mathbb{F}_3$ has the Haagerup property as a group, this carries over to the inclusions $L^\infty(\mathbb{T}^2) \subset M_S$ and $L^\infty(\mathbb{T}^2) \subset M_{S'}$.

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it follows that the inclusion $L^\infty(\mathbb{T}^2) \subset M_S$ has the relative property (T). The same holds for $L^\infty(\mathbb{T}^2) \subset M_{S'}$.

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Thus by Popa's Theorem, any isomorphism between M_S and $M_{S'}$ must carry $L^\infty(\mathbb{T}^2)$ to itself, after possibly conjugating with a unitary.

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are isomorphic, so by Feldman-Moore, the actions σ_S and $\sigma_{S'}$ are orbit equivalent.Thus $\sim_{oe}$ is Borel reducible to isomorphism of factors, and so the isomorphism relation for factors is not classifiable by countable structures.

Proof of Theorem 1

The factors M_S are all of type II_1. One may now proceed to deduce the result for type II_∞ factors by showing that

$$S \mapsto M_S \otimes \mathcal{B}(\ell^2(\mathbb{N})),$$

where $\mathcal{B}(\ell^2(\mathbb{N}))$ denotes the bounded operators on $\ell^2(\mathbb{N})$, is a Borel reduction of $\sim_{oe}$ to $\simeq^{\mathrm{II}_\infty}$.

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Theorem 3

Recall that if $\mathcal{L}$ is a countable language, $\mathrm{Mod}(\mathcal{L})$ denotes the Polish space of models of $\mathcal{L}$ with universe $\mathbb{N}$. $\simeq^{\mathrm{Mod}(\mathcal{L})}$ denotes the isomorphism relation in $\mathrm{Mod}(\mathcal{L})$

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I will now sketch the proof of:

Theorem (S.-T ornquist, '08)

If $\mathcal{L}$ is a countable language then $\simeq^{\mathrm{Mod}(\mathcal{L})} \leq_B \simeq^{\mathcal{F}_{\mathrm{II}_1}(\mathcal{H})}$.

A strong rigidity theorem for Bernoulli shifts

The proof is based on the following rigidity theorem of Popa, which shows that for a Bernoulli shift β coming from certain kind of group, the group can be recovered from the isomorphism type of the group measure space factor $L^\infty(X^G) \rtimes_\beta G$:

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Theorem (Popa, '06)

Suppose G_1 and G_2 are countably infinite discrete groups, β_1 and β_2 are the corresponding Bernoulli shifts on $X_1 = [0,1]^{G_1}$ and $X_2 = [0,1]^{G_2}$, respectively, and $M_1 = L^2(X_1) \rtimes_{\beta_1} G_1$ and $M_2 = L^2(X_2) \rtimes_{\beta_2} G_2$ are the corresponding group-measure space II_1 factors. Suppose further that G_1 and G_2 are ICC (infinite conjugacy class) groups having the relative property (T) over an infinite normal subgroup. Then $M_1 \simeq M_2$ iff $G_1 \simeq G_2$.

Isomorphism of relative property (T) groups

An example of an ICC group with property (T) is $SL(3, \mathbb{Z})$. Any group of the form $H \times SL(3, \mathbb{Z})$ has the relative property (T) (over SL$(3, \mathbb{Z})$). If H is ICC, then $H \times SL(3, \mathbb{Z})$ is ICC.

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Denote by $\mathbf{wT}_{\mathrm{ICC}}$ the class of countable groups, having the relative property (T) over some infinite normal subgroup, and $\simeq^{\mathbf{wT}_{\mathrm{ICC}}}$ the isomorphism relation in that class.

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In other words: $\simeq^{\mathbf{wT}_{\mathrm{ICC}}}$ is *Borel complete* for countable structures, in the sense of Friedman and Stanley.

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Mekler then defines from a given countable nice graph Γ (and a prime p, which we shall keep fixed here) a countable group $G(\Gamma)$, which we will call the *Mekler group* of Γ, and shows that for nice graphs, $\Gamma_1 \simeq \Gamma_2$ iff $G(\Gamma_1) \simeq G(\Gamma_2)$. The association $\Gamma \mapsto G(\Gamma)$ is Borel, and moreover, *for every graph automorphism of* Γ *there is a corresponding group automorphism of* $G(\Gamma)$. However these groups are not ICC.

Definition of Mekler groups

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The *Mekler group* of Γ, denoted $G(\Gamma)$, is defined as

$$\Big(\underset{v\in\Gamma}{\mathbf{2}}\mathbb{Z}/p\mathbb{Z}\Big)/N$$

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$$N = \langle [v_1, v_2] : v_1 \Gamma v_2 \rangle$$

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However, the groups $G(\Gamma)$ are not ICC.

A variant of Mekler's construction

To remedy this, we consider for each connected nice graph Γ with vertex set $\mathbb{N}$ the nice graph $\Gamma_{\mathbb{F}_2}$ with vertex set $\mathbb{N} \times \mathbb{F}_2$ defined by

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Clearly, $\mathbb{F}_2$ acts by graph automorphisms on $\Gamma_{\mathbb{F}_2}$. Going to the corresponding Mekler group $G(\Gamma_{\mathbb{F}_2})$, we have a corresponding action of $\mathbb{F}_2$ by group automorphisms on $G(\Gamma_{\mathbb{F}_2})$. Thus we may form the semi-direct product $G(\Gamma_{\mathbb{F}_2}) \rtimes \mathbb{F}_2$. This group is easily seen to be ICC.

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We now consider the group

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Claim: The map $\Gamma \mapsto G_\Gamma$ is a Borel reduction of isomorphism of nice connected graphs to $\simeq^{\mathbf{wT}_{\text{ICC}}}$.

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$$\simeq^{\mathrm{Mod}(\mathcal{L})} \leq_B \text{iso of connected nice graphs } \leq_B \simeq^{\mathbf{wT}_{\mathrm{ICC}}} \leq_B \simeq^{\mathscr{F}_{\mathrm{II}_1}}$$

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The isomorphism relation for separable von Neumann factors of type II_1 *is complete analytic as a subset of* $\mathscr{F}_{\mathrm{II}_1} \times \mathscr{F}_{\mathrm{II}_1}$.

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Proof.

Since the isomorphism relation $\simeq^{\mathscr{G}}$ of countable graphs, say, is complete analytic and by the previous Theorem $\simeq^{\mathscr{G}} \leq_B \simeq^{\mathscr{F}_{\mathrm{II}_1}}$. □

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  This in turn implies that isomorphism of factors is *not* a universal analytic equivalence relations, i.e. it is not of maximal complexity in the $\leq_B$ hierarchy of analytic equivalence relations.

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